

AN ORDER EIGHT GAUSS QUADRATURE RUNGE – KUTA METHOD FOR ORDINARY DIFFERENTIAL EQUATIONS (ODEs)

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Abstract

We present a strong convergence implicit Runge-Kutta method, with four stages, for solution of initial value problem of first ordinary differential equations. Collocation method is used to derive a continuous scheme which evaluated at special points, the Gaussian points of fourth degree Legendre polynomial, gives us four functional evaluations in the Runge-Kutta method for the iteration of the solutions. Convergent property of the method is discussed. Experimental problems used to check the quality of the scheme show that the method is highly efficient, A – stable, has simple structure, converges to exact solution faster and better than some existing popular methods cited in this paper.

Key words: Strong convergent, Collocation method, Gaussian points, Highly efficient, A – stable, Simple structure,

1.0 Introduction

A number of practical and real-life problems are models of ordinary differential equations (ODEs). Most of these are non-linear, stiff or oscillatory problems. Examples of such are found in Electric circuits systems, chemical reaction problems, population growth, Biology, Social Sciences etc. Explicit Runge-Kutta methods cannot efficiently handled them as they lack sufficient stability region. The instability of explicit Runge-Kutta (RK) methods motivates the development of implicit methods. The coefficient of explicit method is a lower triangular matrix and that of the implicit is a square matrix. We have different types of



implicit methods; Radau, Lobatto, Gauss Legendre quadrature. Butcher (2013) developed Gauss-quadrature method of order four and six respectively. In order to get higher and more stable scheme, we obtain four Gaussian points from the fourth term of Legendre polynomial to derive a simple four stage Gauss quadrature RK method of order 8.

The Legendre polynomial of degree four is $P(x) = \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8}$. The zeros are $-\frac{3\sqrt{206}}{50}, -\frac{17}{50}, \frac{17}{50}, \frac{3\sqrt{206}}{50}$ and its corresponding Gaussian points in $[0,1]$ defined by a transformation. $T(x_i) = \frac{1}{2}[a(x_i - 1) + b(x_i + 1)]$, see Laurie, Duk P. (2001)

In particular $[a, b] = [0,1]$; are $\frac{1}{2} - \frac{3\sqrt{206}}{100}, \frac{33}{100}, \frac{67}{100}, \frac{1}{2} + \frac{3\sqrt{206}}{100}$

1.1 Basic Definitions/Preliminaries:

(i) A Runge-Kutta implicit scheme for first order ODEs is defined as

$$\left. \begin{aligned} Y_{n+1} &= y_n + h \sum_{i=1}^5 b_i F_i, \quad i = 1, 2, \dots, S \quad (S = \text{stages}) \\ \text{where } F_i &= f(x_n + c_i h, y_i), \quad y_i = y_n + \sum_{j=1}^5 a_{ij} f_j \end{aligned} \right\} \quad 1.0$$

(ii) Order of scheme: the order of implicit Runge-Kutta methods whose abscissae or integration. The nodes are Gaussian points of Legendre polynomial of order $p = (2S)$

(iii) Error /error constant: A Runge-Kutta solution can be expanded into Taylor's series, the solution agrees with the Taylor's series expander up to order P , the truncation error is the $O(h^{p+1})$ term. The error and error constant is associated with a linear difference operator $L(y(x), h)$. as $L(y(x), h) = y(x_{n+1}) - y_{n+1}$

$$= C_0 y(x_n) + C_1 h y'(x_n) + (2h^2 y''(x_n) + \dots + C_p h^p y^{(p)}(x_n) + C_{p+1} h^{(n+1)} y(x_n))$$

where $y(x_{n+1})$ and y_{n+1} are exact and Runge – Kutta approximate solutions, respectively.



According to Lambert (1973) the method is of order P, with error constant C_{p+1} of =

$$C_1 = C_0 = C_2 \dots C_p = 0 \text{ and } C_{p+1} \neq 0.$$

(iv) In Gollub et al (1969) The three stage implicit Gauss-quadrature of order 6 is summarized by the table below:

Table 1A. Buchers Coefficient table (1)

C	$A = (a_{ij})$		
$C_1 = \left(\frac{1}{2} - \frac{\sqrt{15}}{10}\right)$	$a_{11} = \frac{5}{36}$	$a_{12} = \left(\frac{2}{9} - \frac{\sqrt{15}}{15}\right)$	$a_{13} = \left(\frac{5}{36} - \frac{\sqrt{15}}{30}\right)$
$C_2 = \left(\frac{1}{2}\right)$	$a_{21} = \frac{5}{36} + \frac{\sqrt{15}}{24}$	$a_{22} = \frac{2}{9}$	$a_{23} = \left(\frac{5}{36} - \frac{\sqrt{15}}{24}\right)$
$C_3 = \left(\frac{1}{2} + \frac{\sqrt{15}}{10}\right)$	$a_{31} = \frac{5}{36} + \frac{\sqrt{15}}{30}$	$a_{32} = \frac{5}{36} + \frac{\sqrt{15}}{15}$	$a_{33} = \frac{5}{36}$
b	$b_1 = \frac{5}{18}$	$b_2 = \frac{4}{9}$	$b_3 = \frac{5}{18}$

General R.K Solution:

$$y_{n+1} = y_n + h \sum_{i=1}^3 b_i F_i$$

$$F_i = f(x + c_i h, Y_i); \quad Y_i = y_n + h \sum_{j=1}^5 a_{ij} F_j \quad (1.1)$$

v. The five stage super convergence multistep collocation method of order 6, by Lie I. and Norsett S.P (1989) is given in the table below



Table 1B:**Butcher's tableau (Lie etal)****C** **A = (a_{ij})**

0	0	0	0	0	0
$\frac{1}{2} - \frac{\sqrt{3}}{4}$	$-\left(\frac{11}{320} - \frac{3\sqrt{3}}{80}\right)$	$\left(\frac{8}{45} - \frac{13\sqrt{3}}{160}\right)$	$\left(\frac{11}{45} - \frac{17\sqrt{3}}{120}\right)$	$\left(\frac{8}{45} - \frac{49\sqrt{3}}{480}\right)$	$-\left(\frac{21}{320} - \frac{3\sqrt{3}}{80}\right)$
$\frac{1}{2}$	$-\frac{7}{40}$	$\left(\frac{8}{45} + \frac{\sqrt{3}}{6}\right)$	$\frac{11}{45}$	$\left(\frac{8}{45} - \frac{\sqrt{3}}{6}\right)$	$\frac{3}{40}$
$\frac{1}{2} + \frac{\sqrt{3}}{4}$	$-\left(\frac{11}{320} + \frac{3\sqrt{3}}{80}\right)$	$\left(\frac{8}{45} + \frac{49\sqrt{3}}{480}\right)$	$\left(\frac{11}{45} + \frac{17\sqrt{3}}{120}\right)$	$\left(\frac{8}{45} + \frac{13\sqrt{3}}{160}\right)$	$-\left(\frac{21}{320} + \frac{3\sqrt{3}}{80}\right)$
1	$-\frac{1}{10}$	$\frac{16}{45}$	$\frac{22}{45}$	$\frac{16}{45}$	$-\frac{1}{10}$
<i>b</i>	$-\frac{1}{10}$	$\frac{16}{45}$	$\frac{22}{45}$	$\frac{16}{45}$	$-\frac{1}{10}$

Gen. Solution

$$y_{n+1} = y_n + h \sum_{i=1}^5 b_i f_i$$

$$F_i = f(x + c_{ih}, y_i), y_i = y_n + h \sum_{j=1}^5 a_{ij} F_j$$

2.0 Methodology

We consider the first order initial value problem of ordinary differential equation.

$$y' = f(x, y), y(x_0) = y_0, \quad a \leq x \leq b. \quad (2.1)$$

We use a Linear multistep method of the form



$$y(x) = \sum_{j=0}^{t-1} \alpha_j(x) y_n + h \sum_{j=1}^{m-1} B_j(x) f(c_j) \quad (2.2)$$

where t are the number of interpolation points x_{n+j} ($j = 0, 1, \dots, t-1$) m is the number of collocation points c_j ($j = 0, 1, \dots, m-1$), $f(x, y)$ is continuously differentiable. The constants α_j, B_j are elements of $(t+m) \times (t+m)$ square matrix. They are selected so that high accurate approximation of the solution of (2.1) is obtained; h is a constant step size.

The $\alpha_j(x), B_j(x)$ in (2.2) can be represented by polynomial of the form.

$$\alpha_j(x) = \sum_{i=0}^{t-1} \alpha_{j,i+1} x^i, \quad i = (0, 1, \dots, t-1)$$

$$B_j(x) = \sum_{i=0}^{m-1} h B_{j,i+1} x^i, \quad i = (0, 1, \dots, m-1) \quad (2.3)$$

The Coefficient $\alpha_{j,i+1}, B_{j,i+1}$ are to be determined. Now substituting (2.3) into (2.2), we have

$$y(x) = \sum_{j=0}^{m+t-1} \left(\sum_{i=0}^{t-1} \alpha_{j,i+1} y_{n+i} + h \sum_{i=0}^{m-1} h B_{j,i+1} f_{n+i} \right) x^i = \sum_{i=0}^{m+t-1} \lambda_j x^i \quad (2.4)$$

where

$$\lambda_j = \left(\sum_{i=0}^{t-1} \alpha_{j,i+1} y_{n+i} + h \sum_{i=0}^{m-1} h B_{j,i+1} f_{n+i} \right), \quad \lambda_j \in R^j, j \in (0, 1, \dots, t+m-1).$$

(2.04) can be exposed in matrix form as $y(x) = (y_n, y_{n+1}, \dots, y_{n+t-1}, f_n, f_{n+1}, \dots, f_{n+m-1}) C^T D$

Where

$$C = \begin{pmatrix} \alpha_{0,1} \dots \alpha_{t-1,1} h B_{0,1} \dots h B_{m-1,1} \\ \alpha_{0,2} \dots \alpha_{t-1,2} h B_{0,2} \dots h B_{m-1,2} \\ \alpha_{0,3} \dots \alpha_{t-1,3} h B_{0,3} \dots h B_{m-1,3} \\ \vdots \end{pmatrix} \quad (2.5)$$



$$\alpha_{0,t+m} \quad \alpha_{t-1, t+m} \quad hB_0, t+m, \dots, hP_{m-1}, t+m$$

$$D = \begin{pmatrix} 1 & x_n & x_n^2 & \dots & x_n^{t+m-1} \\ 0 & 1 & 2x_{n+q_1} & \dots & (t+m-1)x_{n+q_1}^{t+m-2} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 1 & 2x_{n+q_{m-1}} & \dots & (t+m-1)x_{n+q_{m-1}}^{t+m-2} \end{pmatrix} \quad (2.6)$$

q_1 are the collocation points.

Theorem 1.0

Let I denote identity matrix of dimension $(t+m)$, matrices C and D are defined by (2.5) and (2.06) respectively, satisfy.

$$(i) \quad DC = I$$

$$(ii) \quad y(x) = \sum_{i=0}^{m+t-1} \lambda_j x^i \quad - \quad (2.7)$$

Proof: (see Onunanyi et al (1994))

Now we assume a power series solution of degree order 4 of the form.

$$y(x) = \sum_{j=0}^4 \lambda_j x^j, \quad y^i(x) = \sum_{j=0}^4 \lambda_j i x^{i-1} \quad (2.8)$$

Interpolate at x_n , and collocate at $x = x_{n+q_i}$, $i = 0, \dots, 4$, yields system of simultaneous equation of the form.

$$\begin{aligned} \lambda_0 + \lambda_1 x_n + \lambda_2 x_n^2 + \lambda_3 x_n^3 + \lambda_4 x_n^4 &= y_n \\ 0 \lambda_1 + 2 \lambda_2 x_{n+q_1} + \dots + 4 \lambda_4 x_{n+q_1}^3 &= f_n + q_1 \end{aligned} \quad (2.9)$$

.



$$0 \lambda_1 2 \lambda_2 x_{n+q_4} + 3 \lambda_3 4 \lambda_4 x_{n+q_4}^3 = f_n + q_4$$

where λ_j are to be determined and (2.9) can be rewritten in matrix form as

$$\begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 \\ 0 & 1 & 2x_{n+q_1} & 3x_{n+q_1}^2 & 4x_{n+q_1}^3 \\ 0 & 1 & 2x_{n+q_2} & 3x_{n+q_2}^2 & 4x_{n+q_2}^3 \\ 0 & 1 & 2x_{n+q_3} & 3x_{n+q_3}^2 & 4x_{n+q_3}^3 \\ 0 & 1 & 2x_{n+q_4} & 3x_{n+q_4}^2 & 4x_{n+q_4}^3 \end{pmatrix} \begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} y_n \\ f_n + q_1 \\ f_n + q_2 \\ f_n + q_3 \\ f_n + q_4 \end{pmatrix} \quad (2.10)$$

i.e

DA = Y, where

$$D = \begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 \\ 0 & 1 & 2x_{n+q_1} & 3x_{n+q_1}^2 & 4x_{n+q_1}^3 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & 2x_{n+q_4} & 3x_{n+q_4}^2 & 4x_{n+q_4}^3 \end{pmatrix} \quad \text{D-matrix}$$

$$A = (\lambda_1 \quad \lambda_2 \quad \lambda_3 \quad \lambda_4)^T, \quad Y = (y_n, f_{n+q_1}, f_{n+q_2}, f_{n+q_3}, f_{n+q_4})^T \quad (2.11)$$

Using maple mathematical software we obtain the continuous scheme of the form.



$$\text{i.e } y(x) = y_n + h(f_{n+q_1} + f_{n+q_2} + f_{n+q_3} + f_{n+q_4})$$

$$\begin{aligned}
 y(x) = y_n &+ \left[\frac{-q_1 q_2 q_4 x}{(-q_4 + q_1)(-q_2 + q_1)(q_3 + q_1)} + \frac{(h^2 q_2 q_4 + h^2 q_4 q_3 + h^2 q_1 q_3 + h^2 q_4 q_3) x^2}{2h^3(-q_3 + q_1)(-q_2 + q_1)(q_3 + q_1)} \right. \\
 &- \left. \frac{(q_2 h + q_3 h + q_4 h) x^3}{3h^3(-q_3 + q_1)(-q_2 + q_1)(-q_3 + q_1)} + \frac{x^4}{4h^4(-q_4 + q_1)(-q_2 + q_1)(-q_3 + q_1)} \right] f_n + q_1 \\
 &+ \left[\frac{q_1 q_3 q_4 x}{(q_2 - q_4)(-q_2 + q_1)(q_2 - q_3)} - \frac{(h_{q_3}^2 q_4 + h_{q_1}^2 q_4 + h_{q_1}^2 q_3) x^2}{2h^3(q_2 - q_3)(-q_2 + q_1)(q_2 - q_4)} \right. \\
 &+ \left. \frac{(q_1 h + q_4 h + q_3 h) x^3}{3h^3(q_1 - q_3)(-q_2 + q_1)(q_2 - q_3)} - \frac{x^4}{4h^3(q_2 - q_4)(-q_2 + q_1)(q_2 - q_3)} \right] f_n + q_2 \\
 &+ \left[\frac{q_1 q_3 q_4 x}{(q_3 - q_4)(q_1 q_2 - q_1 q_3 + q_3^2 - q_1 q_3)} + \frac{(h_{q_1}^2 q_4 + h_{q_1}^2 q_2 + h_{q_2}^2 q_4) x^2}{2h^3(q_1 q_2 - q_1 q_3 + q_3^2 - q_2 q_3)(q_3 - q_4)} \right. \\
 &- \left. \frac{(q_2 h + q_4 h + q_1 h) x^3}{3h^3(q_3 - q_4)(q_1 q_2 - q_1 q_3 + q_3^2 - q_2 q_3)} + \frac{x^4}{4h^3(q_3 - q_4)(q_1 q_2 - q_1 q_3 + q_3^2 - q_2 q_3)} \right] f_n + q_3 \\
 &+ \left[\frac{q_1 q_3 q_4 x}{q_1 q_2 q_3 - q_1 q_2 q_4 + q_4^2 q_1 - q_1 q_3 q_4 + q_4^2 q_2 - q_2 q_3 q_4 - q_4^3 + q_4^2 q_3} - \right. \\
 &\frac{(h^2 q_1 q_2 + h^2 q_1 q_3 + h^2 q_2 q_3) x^2}{2h^3(q_1 q_2 q_3 - q_1 q_2 q_4 + q_4^2 q_1 - q_1 q_3 q_4 + q_4^2 q_2 - q_2 q_3 q_4 - q_4^3 + q_4^2 q_3)} + \\
 &\frac{(q_1 h + q_2 h + q_3 h) x^3}{3h^3(q_1 q_2 q_3 - q_1 q_2 q_4 + q_4^2 q_1 - q_1 q_3 q_4 + q_4^2 q_2 - q_2 q_3 q_4 - q_4^3 + q_4^2 q_3)} - \\
 &\left. \frac{x^4}{4h^3(q_1 q_2 q_3 - q_1 q_2 q_4 + q_4^2 q_1 - q_1 q_3 q_4 + q_4^2 q_2 - q_2 q_3 q_4 - q_4^3 + q_4^2 q_3)} \right] f_n + q_4
 \end{aligned} \tag{2.12}$$

Now evaluating the continuous at $x = x_{n+j}$ $j = q_1, q_2, q_3$ and q_4 , specifically $q_1 =$

$$\left(\frac{1}{2} - \frac{3\sqrt{206}}{100}\right), q_2 = \frac{33}{100}, q_3 = \frac{67}{100}, q_4 = \left(\frac{1}{2} + \frac{3\sqrt{206}}{100}\right),$$

We obtain the following discrete schemes:

$$\begin{aligned}
 y_{n+q_1} = y_n &+ \left(\frac{1633}{18780} - \frac{71\sqrt{206}}{96717000}\right) h f_{n+q_1} + \left(\frac{134689}{939000} - \frac{927\sqrt{206}}{78250}\right) h f_{n+q_2} \\
 &+ \left(\frac{171511}{939000} - \frac{927\sqrt{206}}{78250}\right) h f_{n+q_3} + \left(\frac{1633}{18780} - \frac{121979\sqrt{206}}{19343400}\right) h f_{n+q_4}
 \end{aligned}$$



$$\begin{aligned}
y_{n+q_2} &= y_n + \left(\frac{7623}{78250} + \frac{1629507\sqrt{206}}{257912000} \right) hf_{n+q_1} + \left(\frac{3470313}{21284000} hf_{n+q_2} - \frac{118701}{4256800} \right) hf_{n+q_3} \\
&+ \left(\frac{7623}{78250} - \frac{1629507\sqrt{206}}{257912000} \right) hf_{n+q_4} \\
y_{n+q_3} &= y_n + \left(\frac{8978}{117375} + \frac{1629507\sqrt{206}}{257912000} \right) hf_{n+q_1} + \frac{4520423}{12770400} hf_{n+q_2} \\
&+ \frac{10410661}{63852000} hf_{n+q_3} + \left(\frac{8978}{117375} - \frac{1629507\sqrt{206}}{257912000} \right) hf_{n+q_4} \\
y_{n+q_4} &= y_n + \left(\frac{1633}{18780} + \frac{121979\sqrt{206}}{19343400} \right) hf_{n+q_1} + \left(\frac{134689}{939000} + \frac{927\sqrt{206}}{78250} \right) hf_{n+q_2} \\
&+ \left(\frac{171511}{939000} - \frac{927\sqrt{206}}{78250} \right) hf_{n+q_3} + \left(\frac{1633}{18780} + \frac{17\sqrt{206}}{96717000} \right) hf_{n+q_4} \quad - (2.13)
\end{aligned}$$

To convert to Runge-Kutta function evaluations, the discrete schemes (2.13) must satisfy (2.01), hence

$$\begin{aligned}
y'_{n+q_1} &= f(x_{n+q_1}, y_{n+q_1}) = f(x_{n+q_1}, y_n + \left(\frac{1633}{18780} - \frac{71\sqrt{206}}{96717000} \right) hf_{n+q_1} \\
&+ \left(\frac{134689}{939000} - \frac{927\sqrt{206}}{78250} \right) hf_{n+q_2} + \left(\frac{171511}{939000} - \frac{927\sqrt{206}}{78250} \right) hf_{n+q_3} + \left(\frac{1633}{18780} - \frac{121979\sqrt{206}}{19343400} \right) hf_{n+q_4} \\
y'_{n+q_2} &= f(x_{n+q_2}, y_{n+q_2}) = f(x_{n+q_2}, y_n + \left(\frac{7623}{78250} + \frac{1629507\sqrt{206}}{257912000} \right) hf_{n+q_1} + \frac{3470313}{21284000} hf_{n+q_2} \\
&- \frac{118701}{4256800} hf_{n+q_3} + \left(\frac{7623}{78250} - \frac{1629507\sqrt{206}}{257912000} \right) hf_{n+q_4}] \\
y'_{n+q_3} &= f(x_{n+q_3}, y_{n+q_3}) = f(x_{n+q_3}, y_n + \left(\frac{8978}{117375} + \frac{1629507\sqrt{206}}{257912000} \right) hf_{n+q_1} + \frac{4520423}{12770400} hf_{n+q_2} \\
&+ \frac{10410661}{63852000} hf_{n+q_3} + \left(\frac{8978}{117375} - \frac{1629507\sqrt{206}}{257912000} \right) hf_{n+q_4}] \quad (2.14) \\
y'_{n+q_4} &= f(x_{n+q_4}, y_{n+q_4}) = f(x_{n+q_4}, y_n + \left(\frac{1633}{18780} + \frac{121979\sqrt{206}}{19343400} \right) hf_{n+q_1} + \left(\frac{927\sqrt{206}}{78250} \right) hf_{n+q_2}
\end{aligned}$$



$$+ \left(\frac{171511}{939000} + \frac{927\sqrt{206}}{78250} \right) h f_{n+q_3} + \left(\frac{1633}{18780} + \frac{71\sqrt{206}}{96717000} \right) h f_{n+q_4}]$$

Putting $y'_{n+q_1} = K_1 = f(x_{n+q_1}, y_{n+q_1})$ or f_{n+q_1} , $K_2 = f(x_{n+q_2}, y_{n+q_2})$ or f_{n+q_2} etc.,

We obtain the function evaluations as

$$\begin{aligned} K_1 &= f\left[x_n + \left(\frac{1}{2} - \frac{3\sqrt{206}}{100}\right)h, y_n + \left(\frac{1633}{18780} - \frac{71\sqrt{206}}{96717000}\right)hk_1 + \left(\frac{134689}{939000} - \frac{927\sqrt{206}}{78250}\right)hk_2 \right. \\ &\quad \left. + \left(\frac{171511}{939000} - \frac{927\sqrt{206}}{78250}\right)hk_3 + \left(\frac{1633}{18780} - \frac{121979\sqrt{206}}{19343400}\right)hk_4\right] \\ K_2 &= f\left[x_n + \frac{33}{100}h, y_n + \left(\frac{7623}{78250} + \frac{1629507\sqrt{206}}{257912000}\right)hk_1 + h\left(\frac{3470313}{21284000}\right)hk_2 - \left(\frac{118701}{4256800}\right)hk_3 \right. \\ &\quad \left. + \left(\frac{7623}{78250} - \frac{1629507\sqrt{206}}{257912000}\right)hk_4\right] \\ K_3 &= f\left[x_n + \frac{67}{100}h, y_n + \left(\frac{8978}{117375} + \frac{1629507\sqrt{206}}{257912000}\right)hk_1 + \left(\frac{4520423}{12770400}\right)hk_2 + \frac{10410661}{63852000}hk_3 \right. \\ &\quad \left. + \left(\frac{8978}{117375} - \frac{1629507\sqrt{206}}{257912000}\right)hk_4\right] \\ K_4 &= f\left[x_n + \left(\frac{1}{2} + \frac{3\sqrt{206}}{100}\right)h, y_n + \left(\frac{1633}{18780} + \frac{121979\sqrt{206}}{19343400}\right)hk_1 + \left(\frac{134689}{939000} + \frac{927\sqrt{206}}{78250}\right)hk_2 \right. \\ &\quad \left. + \left(\frac{171511}{939000} + \frac{927\sqrt{206}}{78250}\right)hk_3 + \left(\frac{1633}{18780} + \frac{71\sqrt{206}}{96717000}\right)hk_4\right] \end{aligned} \quad (2.15)$$

The weight is $b = (b_1, b_2, b_3, b_4)$; Evaluating the continuous scheme (2.12) at $x = 1$ we obtain

$$b = \left(\frac{1633}{9390}, \frac{1531}{4695}, \frac{1531}{4695}, \frac{1633}{9390} \right) \quad (2.16)$$

The four stage Runge-Kutta method is defined as

$$y_{n+1} = y_n + h \sum_{i=1}^4 b_i k_i = y_n + \frac{1633}{9390}h (K_1 + K_4) + \frac{1531}{4695}h (K_2 + K_3)$$

Where K_i $i = 1, 2, \dots, 3$ are given by (2.15)



3.00 Analysis of the scheme.

We can be write (2.15) in Butchers tableau as:

C	$A = (a_{ij}), \quad i, j = 1, 2, \dots 4.$			
$\frac{1}{2} - \frac{3\sqrt{206}}{100}$	$\left(\frac{1633}{18780} - \frac{71\sqrt{206}}{96717000}\right)$	$\left(\frac{134689}{939000} - \frac{927\sqrt{206}}{78250}\right)$	$\left(\frac{171511}{939000} - \frac{927\sqrt{206}}{78250}\right)$	$\left(\frac{1633}{18780} - \frac{121979\sqrt{206}}{19343400}\right)$
$\frac{33}{100}$	$\left(\frac{7623}{78250} - \frac{1629507\sqrt{206}}{257912000}\right)$	$\frac{347013}{21284000}$	$-\frac{118701}{4256800}$	$\left(\frac{7623}{78250} + \frac{1629507\sqrt{206}}{257912000}\right)$
$\frac{67}{100}$	$\left(\frac{8978}{117375} + \frac{1629507\sqrt{206}}{257912000}\right)$	$\frac{4520423}{12770400}$	$\frac{10410661}{63852000}$	$\left(\frac{8978}{117375} + \frac{1629507\sqrt{206}}{257912000}\right)$
$\frac{1}{2} + \frac{3\sqrt{206}}{100}$	$\left(\frac{1633}{18780} + \frac{121979\sqrt{206}}{19343400}\right)$	$\left(\frac{134689}{939000} + \frac{927\sqrt{206}}{78250}\right)$	$\left(\frac{171511}{939000} + \frac{927\sqrt{206}}{78250}\right)$	$\left(\frac{1633}{18780} + \frac{71\sqrt{206}}{96717000}\right)$
b	$\frac{1633}{9390}$	$\frac{1531}{4695}$	$\frac{1531}{4695}$	$\frac{1633}{9390}$

Or can be summarized as

C	A	U
	B	V

Where $(C = C_1 C_2 \dots C_4)^T$ $A = (a_{ij}) \quad y = 1, 2, \dots 4,$ $U = (1, 1, 1, 1)^T$ $V = (1)$

(i) Consistency:

The Runge-Kutta method is consistent since

$$\sum_{j=1}^4 a_{ij} = C_i, \quad \sum_{j=1}^4 b_j = 1. \quad (\text{see Table 2.16})$$



(ii) Stability:

The stability of the method is investigated by considering the linear test equation.

$$y' = \lambda y, \quad \lambda \in \mathbb{C}$$

Putting $Z = \lambda h, \quad h \in (0, 1)$

The stability function is $R(Z)$

$$R(Z) = I J + Z b^T (I - Z A) - 1$$

Where I is the identity matrix

$b = (b_1, b_2, b_3, b_4)$, the weight, $e = (1, 1, 1, 1)$, A is the Runge – Kutta matrix of Coefficients (2.16).

The region of A- stability is the set of point satisfying $R(Z) = \{ Z: R(Z) \leq 0 \text{ and } \operatorname{Re}(Z) \leq 0 \}$

The characteristic polynomial is defined as

$P(Z) = \det(R(Z) - I \lambda)$, which can be potted using maple or math lap, to determine the region of A- stability. see (K Burrage and JC Butcher (1979))

(iii) Order and error of the scheme:

The order of the method is $p = 2S = 8$, since the nodes or absysaes are Gaussian points of fourth term Gauss – Legendre Polynomial and all Gauss- quadrantine methods have order $P = 2S$, (S = stages) see Gollub et al (1969)

The Runge – Kutta solution y_{n+1} can be expanded into Taylors series as

$$\begin{aligned} y_{n+1} &= y_n h \sum_{i=1}^4 b_i K_i = y_n + h \sum_{i=1}^4 b_i f(x_{n+q_1}, y_n + q_1) \\ &= y_n + h \sum_{i=1}^4 b_i y'_{n+q_1} \\ &= y_n + C_1 h y'_n + C_2 h^2 y''_n + \dots + C h^p y_{n(p+1)} \end{aligned}$$



The R.K solution agrees with Taylor's series expansion up to term in h^8 . The truncation error is $O(h^9)$.

Numerical Experiments / Results:

To show the efficiency of this new method, we use three problems to compare our computational solutions with exact and other similar methods.

Problem I. (Electric circuit problem model)

In an Electric RL – circuit system, L is the inductance constant, I the current, R the resistance constant, and $E(t)$, periodic electromotive force. Find the currents at time $t = 0.2$ (0.2) 0.1 secondly; with $L = .1$, $R = 5$, $E(t) = \sin(t)$, with initial condition $I(0) = 1.0$.

Model: By Kirchhoff's voltage law (KVL) the voltage across the resistor is RI , the voltage drop across inductor is E_L . By KVL Law the sum of two voltage must be equal to electromotive force $E(t)$.

Thus our problem reduces to differential equation $L \frac{dI}{dt} + RI = E(t)$

i.e. $\frac{dI}{dt} = -50 I + 10 \sin(t)$, $y(0) = 1.0$.

Example 2: (Logistic Population Model)

A logistic mathematic model has useful application to human population and animal populations []. The mathematical model is defined as

$$y' = Ay(t) - B y^2(t), \quad y(t_0) = y_{01} \quad A, B > 0.$$

If $y(0) < \frac{A}{B}$, the population is monotonic increasing to limit $\frac{A}{B}$ and if $y(0) > \frac{A}{B}$, it decreases to limit $\frac{A}{B}$.

Now we consider a logistic problem



$$y' = Ay(t) - B y^2(t), \quad y(t_0) = y_0 \quad \text{with } A = 5 \quad B = 3 \quad \text{initial condition} \quad y(0) =$$

2 Million Cows in Kaduna State, Nigeria, t is measured in months.

Model: The logistic problem reduces to differential

Equation

$$y' = 5y - 3y^2, \quad y(0) = 2, \quad h = .2$$

Example 3: (Chemical Reaction Model):

A bimolecular reactions a and B combines $a \in A$ moles per litre of distance A and $b \in B$ moles of substance B given $y(t)$ is the number of moles per litre that have reacted after time t we can find the number of moles per litre which reacted at given times.

Model: We use chemical reaction Law which states that under constant temperature, the chemical reaction is proportional to the product of concentrations of the reacting substances, thus the model can be defined by a differential equation:

$$y' = K(a - y)(b - y), \quad y(t_0) = y_0, \quad K \text{ is the proportional constant, } a, b \text{ are constants.}$$

In our example we consider $a = 5, b = 3, y(0) = 0$ and $K = 1$. Thus our model reduce to an initial value problem $y' = y^2 - 8y + 15, \quad y(0) = 0$.

We can compute the solutions at $t = 0(.1).5$.

The following are the solution tables for comparison.

Notations:

$y(x_{n+1}), n = 0, 1, \dots$ = Exact solutions, with step – size h .

y_{n+1} = The new R.K approximate solutions.

New Method = $|y(x_{n+1}) - y_{n+1}|$, absolute error.

ER_1 = absolute error of – point Gauss – quadrature (3)method.



Table 1: Comparison of numerical solution of problem I. (RI- Circuit). Analytic solution,

$$y(t) = \frac{500}{2501} \sin(t) + \frac{2511}{2501} e^{-50t} - \frac{10}{2501} \cos(t)$$

t	$y(t)$	y_{n+1}	ER_1	ER_2	New Method
.02	0.3693509037	0.3693509135	3.81 E-06	3.81 E-06	9.87 E-09
.04	0.1398758745	0.1398758818	2.80 E-06	1.57 E-06	7.27 E-09
.06	0.0579829374	0.0579829414	1.55 E-06	8.65 E-07	4.01 E-09
.08	0.0303798097	0.0303798097	7.58 E-07	4.24 E-07	1.97 E-09
.10	0.2274516347	0.2274516347	3.49 E-07	1.95 E-07	9.04-10

Table 2: Comparison of Numerical solution of problem 2.

Analytic Solution: $y(t) = -\frac{10}{-6+e^{-st}}$

t	$y(t)$	y_{n+1}	ER_1	ER_2	New Method
.2	1.7755301747	1.7755301873	3.37 E-06	2.13 E-06	1.25 E-08
.4	1.7051273147	1.7051273208	1.74 E-06	1.57 E-06	6.12 E-09
.6	1.6806121251	1.6806121278	7.92 E-07	8.65 E-07	2.66 E-09
.8	1.6717699224	1.6717699235	3.44 E-07	4.24 E-07	1.12 E-09
.10	1.6685404228	1.6685404233	1.46 E-07	1.95 E-07	4.62 E-10

Table 3: Comparison solution of problem 3.

Analytic Solution: $y(t) = \frac{15(e^{2t}-1)}{5e^{-2t}-3}$

t	$y(t)$	y_{n+1}	ER_1	ER_2	New Method
.1	1.0688853015	1.0688853144	9.93 E-08	5.18 E-06	1.30 E-08
.2	1.6544440817	1.6544440919	7.72 E-08	3.66 E-06	1.02 E-08
.3	2.0180987318	2.0180987391	5.55 E-08	2.49 E-06	7.31 E-09
.4	2.2617841981	2.2617842034	4.03 E-08	1.17 E-06	5.28 E-09



.5	2.4335031421s	2.4335031460	2.98 E-08	1.26 E-06	3.89 E-09
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Discussion / Conclusion

The new method has four stages compared with Lie and Norsett (1989) with five stages thus the implementation cost is cheaper. The new method converges better and faster to the exact solutions (see comparison tables above), also it has larger area of A-stability than various methods compared in this research , consequently we have developed a new Gauss- Quadrature Runge-Kutta method with four points or stages only.

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