EFFECT OF TRANSIENT AND FREQUENCY RESPONSE ON RC AND RLC CIRCUITS TO INSTANTANEOUS FORCING FUNCTION

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Abstract
This study examines the effect transient and frequency response on RC and RLC circuits to Instantaneous forcing functions. The governing equations of electrical vibration for both circuits were formulated using Kirchhoff's voltage law. The Laplace transform method was employed to find the transfer functions of both circuits giving solutions as frequency response and transient response respectively. It was observed that for transient response of both RC and RLC circuits, a voltage surge occurs immediately after t=0 and this phenomenon repeats after every second. While for the frequency response of both circuits, it was observed that for low-pass voltage, frequencies lower than 2000Hz are required and for high-pass voltage, frequencies higher than 10^4Hzare needed for optimal functionality.

Keywords: Transient, Frequency, Instantaneous, Vibration, Transfer, Circuits.

1.0 Introduction
Equations of electrical vibrations are very important in understanding the behaviour of electrical circuits especially, where there is a tendency for a voltage surge which can in turn damage electrical components and this has prompted a lot of studies on electrical circuits. Lee and Ormsby (1992) presented a new model for the qualitative analysis of electrical circuit behaviour. They showed that a qualitative representation of electrical resistance provided a good intuitive model of connectivity. Harwood (2011) modelled an RLC circuit's current with differential equations. The circuit was powered by a solar source that had its output voltage passed through an inverter to produce an AC output signal which then made the voltage to become a sinusoidal function of time. Creighton (2011) presented an overview of the Laplace transform along with its

application to several well-known electrical circuits. He focused on systems containing discontinuous forcing terms. Each analytical solution was tested empirically against the actual behaviour of the circuit. In another light, Sheikh (2012) estimated the optimal time step and compared it with the ODE solver of Matlab package of RLC circuit using numerical methods. In order to achieve this, a table was constructed for the model to evaluate optimal time step and also CPU time into the simulation using Matlab 7.6.0 (R2008a). Manuel, Jose and Juan (2014) carried out a research on fractional RC and LC electrical circuits whose derivatives are of the Caputo type. The order of the derivative considered was$0 < \gamma < 1$. To keep the dimensionality of the physical parameters R, L, C, the new parameterσ was introduced. Omijeh and Ogboukebe (2015) investigated the performance of a transfer function characteristic of an RLC circuit, asserted that the method of analysing RLC circuits was never constant and since the resistor, inductor and the capacitor were used in every electronic system, a proper understanding of the system was necessary to know what happened to the system when any parameter is altered.

In these studies however, the forcing functions are not instantaneous. Instantaneous forcing functions known as the Impulse or Dirac delta functions represent the voltage surge in the circuit. Thus, this paper examines the transient and frequency response of RC and RLC circuits to instantaneous forcing function. The governing equations of electrical vibration for both circuits are formulated using Kirchhoff's voltage law. The Laplace transform method is employed to find the transfer functions of both circuits giving solutions as frequency response and transient response respectively.Each analytical solution is tested empirically against the actual behaviour of RC and RLC circuits.

2.0 Governing equation

The governing equations for both RC and RLC circuits are respectively given as:

$$e(t) = R\,q'(t) + \frac{1}{C}q(t), \qquad q(0) = 0 \qquad (1)$$

$$e(t) = L\,\frac{d^2q(t)}{dt^2} + R\,\frac{dq(t)}{dt} + \frac{1}{c}q(t), \qquad q(0) = 0, \quad q'(0) = 0 \qquad (2)$$

where, $e(t)$ is a rectangular pulse wave representing the voltage source, $q(t)$ is the quantity of charge, t is time given in seconds, L is the inductance of the inductor, R is the resistance of the resistor and C is the capacitance of the capacitor.

3.0 Method of Solution

To obtain the transient and frequency response, Laplace transform method of solution is applied. The Laplace transformation is a very powerful tool for solving linear differential equations with

constant coefficients encountered in the study of engineering problems. The main advantage of Laplace transform method is that it automatically takes care of the initial conditions and the direct solution of non-homogeneous differential equations is possible.
Mathematically, it is defined

$$F(s) = L\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt \qquad (3)$$

where "t" is a real number and "s" is a complex number.
while, the Inverse Laplace transform can be defined as a transform that permits us to go from complex domain "s" to time domain "t". It is expressed as;

$$L^{-1}[F(S)] = f(t) = \int_0^\infty F(S)e^{st}ds \qquad (4)$$

We will need to define a window function here; a window function is a mathematical function that has a zero-value outside of some chosen interval or it is a function that is zero outside of its window. Hence the window function is defined as;

$$w(t) = \begin{cases} f(t), & 0 \le t < T \\ 0, & T \le t < \infty, t < O \end{cases} \qquad (5)$$

To get the Laplace transform of the window function$w(t)$, we need to rewrite $f(t)$ as a sum of translated window functions.

$$f(t) = w(t) + w(t-T) + w(t-2T) + \cdots + = \sum_{k=0}^{\infty} w(t-kT) \qquad (6)$$

Since the window function is 0 outside of its window,then

$$H(t-kT) = 1 \qquad (7)$$

where $H(t-kT)$ is the Heaviside unit step function defined as,

$$H(t-kT) = \begin{cases} 0, & t < kT \\ 1, & t \ge kT \end{cases} \qquad (8)$$

and

$$H(t-kT)w(t-KT) = w(t-kT) \qquad (9)$$

Thus,

$$f(t) = \sum_{k=0}^{\infty} H(t-kT)w(t-kT) \tag{10}$$

We now take Laplace transform of $f(t)$, in equation (10)

$$L\,[f(t)] = \sum_{k=0}^{\infty} L[H(t-kT)w(t-kT)] \tag{11}$$

this implies that,

$$F(s) = \sum_{k=0}^{\infty} \int_0^{\infty} H(t-kT)w(t-kT)e^{-st}dt \tag{12}$$

To carry out this integration, we will use separation of variables. Let,

$$x = t - kT,\ \ then\ \ t = x + kT\ \ \ and\ \ \ dt = dx \tag{13}$$

We substitute equation (13) in equation (12) to obtain,

$$F(s) = \sum_{k=0}^{\infty} \int_0^{\infty} H(x-kT+kT)w(x-kT+kT)e^{-s(x+KT)dx} \tag{14}$$

or

$$F(s) = \sum_{k=0}^{\infty} \int_0^{\infty} H(x)w(x)e^{-s(x+kT)}\,dx \qquad ,x \geq 0, H(x) = 1 \tag{15}$$

Then,

$$F(s) = \sum_{k=0}^{\infty} \int_0^{\infty} 1 * w(x)e^{-sx}\,e^{-skT}dx \tag{16}$$

or

$$F(s) = \sum_{k=0}^{\infty} e^{-skT}w(s) \tag{17}$$

Since,

$$\int_0^{\infty} W(x)e^{-sx}dx = W(s) \tag{18}$$

The summation in equation (17) is a geometric series, where$e^{-skT} < 1$ and thus, it converges to:

$$\sum_{k=0}^{\infty}(e^{-skT}) = \frac{1}{1 - e^{-sT}} \tag{19}$$

With (19), equation (17) becomes,

$$F(s) = w(s)\sum_{k=0}^{\infty}(e^{-skT}) \tag{20}$$

Thus,

$$F(s) = w(s) \times \frac{1}{1 - e^{-sT}} \tag{21}$$

4.0 Solution of the governing equation

We begin by getting the transient and frequency response of the RC circuit followed by that of the RLC circuit.

4.1 Transient response of the RC circuit

By taking the Laplace transform of equation (1), we get;

$$e(s) = R(sq(s) - q(0)) + \frac{1}{C}q(s) \tag{22}$$

Applying the initial condition $q(0) = 0,$ we have;

$$e(s) = Rsq(s) + \frac{1}{C}q(s) \tag{23}$$

Rearranging equation (23) we obtain,

$$q(s) = \frac{e(s)}{Rs + \frac{1}{C}} \tag{24}$$

We now need to find $e(s)$ which the Laplace transform isof$e(t)$. To do this, we have to find the Laplace transform of its window $w(t)$which is given as;

$$w(t) = A[\delta(t)] \tag{25}$$

where A is the amplitude of the rectangular pulse wave with period one and $\delta(t)$ is the Dirac delta function defined as

$$\delta(t) = \begin{cases} 0, & t \neq 0 \\ \infty, & t = 0 \end{cases} \tag{26}$$

Now,

$$L[w(t)] = AL[\delta(t)] \tag{27}$$

That is,

$$L[w(t)] = A \int_0^\infty e^{-st}\, \delta(t) dt \tag{28}$$

We will evaluate the integration by parts;

$$A \int_0^\infty e^{-st}\, \delta(t) dt = A\left(e^{-st} H(t)\Big|_0^\infty + \int_0^\infty s e^{-st}\, H(t) dt\right)$$

$$= A\left(0 + s\int_0^\infty e^{-st}\, H(t) dt\right) \tag{29}$$

$$= As\frac{1}{s} \tag{30}$$

or,

$$L[w(t)] = A \tag{31}$$

where,

$$H(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases} (32)$$

We note here that if w(s) is the Laplace transform of a window of some function with period T, then from equation (21), the Laplace transform of the function is,

$$e(s) = \frac{w(s)}{1 - e^{-s}} \tag{33}$$

and equation (24) will then become,

$$q(s) = \frac{1}{Rs + \frac{1}{C}} \times \frac{A}{1 - e^{-s}} \tag{34}$$

Simplifying equation (34) we have,

$$q(s) = \left(\frac{\frac{1}{R}}{s + \frac{1}{RC}} \right) \times \frac{A}{1 - e^{-s}} \tag{35}$$

further simplification yields,

$$q(s) = \frac{A}{R} \left(\frac{1}{s + \frac{1}{Rc}} \right) \times \frac{1}{1 - \ e^{-s}} \tag{36}$$

or,

$$q(s) = \frac{A}{R} \left(\frac{1}{S + \frac{1}{RC}} \right) \sum_{k=0}^{\infty} e^{-ks} \ , provided \ \ e^{-S} < 1 \tag{37}$$

which implies,

$$q(s) = \sum_{k=0}^{\infty} e^{-ks} \left(\frac{\frac{A}{R}}{S + \frac{1}{RC}} \right) \tag{38}$$

Taking inverse Laplace transform of equation (38) we have,

$$L^{-1}[q(s)] = \ L^{-1} \left[\frac{A}{R} \sum_{k=0}^{\infty} \left(\frac{1}{s + \frac{1}{RC}} \right) \times e^{-kS} \right] \tag{39}$$

and by second shifting theorem,

$$q(t) = \frac{A}{R} \sum_{k=0}^{\infty} e^{-\frac{(t-k)}{RC}} H(t-k) \tag{40}$$

Equation (40) gives the transient response of the RC circuit.

The key point is that for all $t < k$, $H(t-k) = 0$. If we are interested in say, the region $t < 4$, we need only calculate up to $k = 3$.

4.2 Frequency response of the Low-pass RC circuit

The transfer function of the output wave of the low-pass voltage is given as,

$$T(j\omega) = \frac{1 - \mathrm{RC}j\omega}{1 + (RC\omega)^2} \tag{41}$$

where ω is the frequency.

We now obtain the magnitude and phase shift of the output wave since the magnitude tells us about the size of the output wave at different frequencies and the phase shift tells us about the angles at which magnitude occurs.

4.2.a The magnitude of the output wave

The magnitude of the output wave is obtained by taking the absolute value of equation (41) i.e.

$$|\mathrm{T(j\omega)}| = \sqrt{\left(\frac{1}{1 + (RC\omega)^2}\right)^2 + \left(\frac{-RC\omega}{1 + (RC\omega)^2}\right)^2} \tag{42}$$

The plot of the magnitude is given as,

$$\text{Magnitude (decibel)} = 10 \log_{10}|\mathrm{T(j\omega)}|^2 \tag{43}$$

4.2.b The phase shift of the output wave

The phase shift of the output wave is the angle of equation (41) given in degrees as,

$$\text{Phase shift} = tan^{-1}\left(\frac{im[T(j\omega)]}{re[T(j\omega)]}\right) \tag{44}$$

or,

$$Phase\ shift = \ tan^{-1}\left(\frac{-RCw}{1+(RC\omega)^2}\times\frac{1+(RC\omega)^2}{1}\right) \tag{45}$$

Thus,

$$\text{Phase shift} = tan^{-1}(-RC\omega) \tag{46}$$

4.3 Frequency response of the high-pass RC circuit

The transfer function of the output wave of a high-pass voltage is given as,

$$T(j\omega) = \frac{(RC\omega)^2 + RCj\omega}{(RC\omega)^2 + 1} \tag{47}$$

and ω is the frequency

4.3.a The Magnitude of the output wave

The magnitude of the output wave is calculated in a similar way to that of the low-pass voltage.

$$|T(j\omega)| = \sqrt{\left(\frac{(RC\omega)^2}{1+(RC\omega)^2}\right)^2 + \left(\frac{RC\omega}{1+(RC\omega)^2}\right)^2} \tag{48}$$

4.3.b Phase shift of the output wave

The phase shift here is also calculated in a similar fashion to that of the low-pass voltage.

$$\text{Phase shift} = tan^{-1}\left(\frac{im[T(j\omega)]}{re[T(j\omega)]}\right) \tag{49}$$

or,

$$\text{Phase shift} = tan^{-1}\left(\frac{1}{RC\omega}\right) \tag{50}$$

4.4 Transient response of the RLC circuit

Applying the Laplace transform to equation (2) we obtain,

$$e(s) = \ Rsq(s) + \frac{q(s)}{C} + \ Ls^2q(s) \tag{51}$$

Rearranging equation (51) we have,

$$q(s) = \frac{e(s)}{Ls^2 + Rs + \frac{1}{C}} \tag{52}$$

We recall that the input voltage $e(t)$ is a rectangular pulse wave having a period of one and defined as,

$$e(t) = A\delta(t) \tag{53}$$

The Laplace transform of equation (53)is,

$$e(s) = \frac{A}{1 - e^{-s}} \tag{54}$$

Using equation (54) in (52) we obtain,

$$q(s) = \frac{A}{1 - e^{-s}} \times \frac{1}{Ls^2 + Rs + \frac{1}{C}} \tag{55}$$

Noting that,

$$\frac{1}{1 - e^{-s}} = \sum_{k=0}^{\infty} e^{-ks} \tag{56}$$

Equation (55) becomes,

$$q(s) = A \sum_{k=0}^{\infty} e^{-ks} \frac{1}{Ls^2 + Rs + \frac{1}{C}} \tag{57}$$

or,

$$q(s) = A \sum_{k=0}^{\infty} e^{-ks} \frac{1}{L\left(s^2 + \frac{Rs}{L} + \frac{1}{LC}\right)} \tag{58}$$

Hence,

$$q(s) = \frac{A}{L} \sum_{k=0}^{\infty} e^{-ks} \frac{1}{\left(s + \frac{R}{2L}\right)^2 + \frac{1}{LC} - \left(\frac{R}{2L}\right)^2} \tag{59}$$

If we let,

$$b^2 = \frac{1}{LC} - \left(\frac{R}{2L}\right)^2 \tag{60}$$

Then we can write equation (59) as,

$$q(s) = \frac{A}{Lb}\sum_{k=0}^{\infty} e^{-ks}\frac{b}{\left(s+\frac{R}{2L}\right)^2 + b^2} \tag{61}$$

Taking the Laplace inverse transform of equation (61) we get,

$$q(t) = \frac{A}{Lb}\sum_{k=0}^{\infty} H(t-k)e^{-\frac{R(t-k)}{2L}}\mathrm{Sinb(t-k)} \tag{62}$$

Equation (62) is the transient response of the RLC circuit.

4.5 Frequency response of the low-pass RLC circuit

The transfer function of the output wave of the low-pass voltage is given as,

$$T(j\omega) = -\frac{1}{LC\omega^2 - RCj\omega - 1} \tag{63}$$

4.5.a The Magnitude of the output wave

The magnitude of the output wave is the absolute value of (63) given as,

$$|T(j\omega)| = \sqrt{\frac{1}{(LC\omega^2 - RC\omega - 1\,)^2}} \tag{64}$$

4.5.b The phase shift of the output wave

The phase shift of the output wave is the angle of equation (63) given in degrees,

$$\text{Phase shift} = tan^{-1}\left(\frac{im[T(j\omega)]}{re[T(j\omega)]}\right) \tag{65}$$

4.6 Frequency response of the high-pass RLC circuit

The transfer function of the output wave of the high-pass voltage is given as,

$$T(j\omega) = \frac{-RCj\omega}{LC\omega^2 - RCj\omega - 1} \tag{66}$$

4.6.a The Magnitude of the output wave

The magnitude of the output wave is given as,

$$|T(j\omega)| = \sqrt{\left(\frac{RC\omega}{LC\omega^2 - RC\omega - 1}\right)^2} \tag{67}$$

4.6.b Phase shift of the output wave

$$\text{Phase shift} = tan^{-1}\left(\frac{im[T(j\omega)]}{re[T(j\omega)]}\right) \tag{68}$$

5.0 Numerical Results and Discussion

To illustrate the analysis presented in this work, we consider for the RC circuit, the following values: $R = 954.14\Omega$, $C = 0.2258\mu F$ and $A = 10V(Voltage\ Source, Peaked\ value)$.
While for the RLC circuit, the values are $R = 0.5999\Omega$, $C = 2262\mu F$, $L = 39.9mH$ and $A = 10V$.
Figure 1 displays the transient response of the RC circuit. It is observed that the highest voltage for this circuit over the time range (0, 6) occurs at time $t = 1$ after which the voltage tends to zero. In figures 2 and 3, the amplitude of the frequency response of the low-pass RC circuit and phase of the frequency response of the same circuit respectively are depicted. It is clearly seen that the cut-off frequency for the low-pass RC circuit is 2000Hz which tallies with -3dB and the phase angle is -14 degrees. Thus, for frequencies higher than 2000Hz, the low-pass RC circuit will cease to function. The amplitude of the frequency response of a high-pass RC circuit and the phase of the frequency response of the same circuit are displayed in figures 4 and 5 respectively. It is shown that the cut-off frequency for the high-pass RC circuit occurs at 8000Hz which corresponds with -3dB and the phase angle is 45 degrees. Hence for the high-pass RC circuit to operate normally, frequencies higher than 8000Hz are required.

In the case of the RLC circuits, the transient response of the circuit is displayed in figure 6. It is seen that a voltage surge occurs immediately after t=0 and this phenomenon repeats after every second. Figures 7 and 8 depict the amplitude of the frequency response of the low-pass RLC circuit and phase of the frequency response of the same circuit respectively. It is observed that the cut-off frequency for the low-pass RLC circuit is 4×10^4Hz with a phase angle of -162 degrees and as such frequencies lower than 4×10^4Hz are required for the low-pass RLC circuit to function optimally. In figures 9 and 10, the amplitude of the frequency response of a high-pass

RLC circuit and the phase of the frequency response of the same circuit are displayed. It is clearly seen that the cut-off frequency occurs at 10^4Hz with a phase angle of -44 degrees and as a result, the high-pass circuit requires higher frequencies than 10^4Hz for it to work optimally.

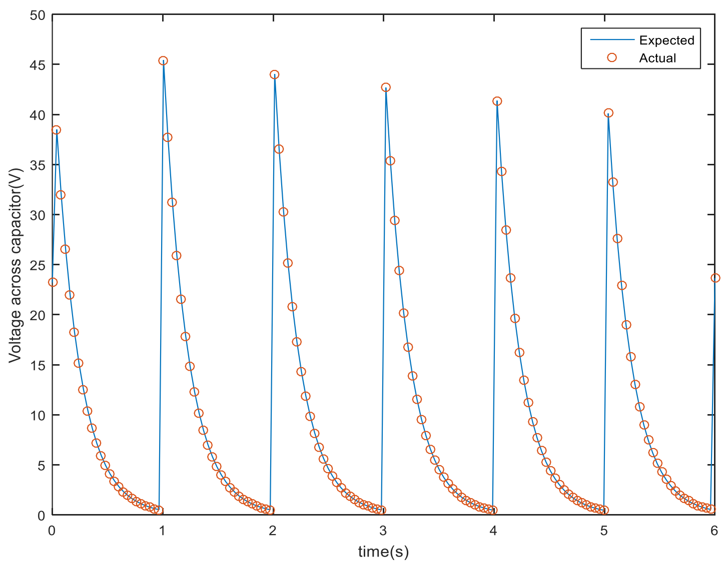

Figure 1: Behaviour of an RC circuit-Transient response

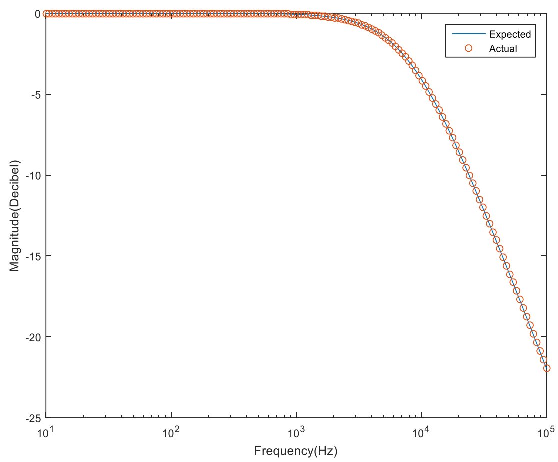

Figure 2: Amplitude of the frequency response of a low-pass RC circuit

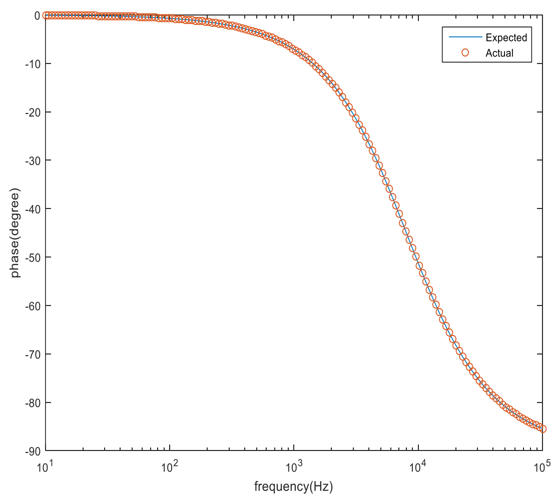

Figure 3: Phase of the frequency response of a low-pass RC circuit

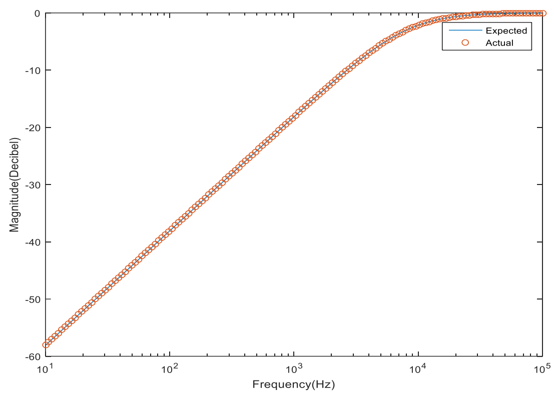

Figure 4: Amplitude of the frequency response of a high-pass RC circuit

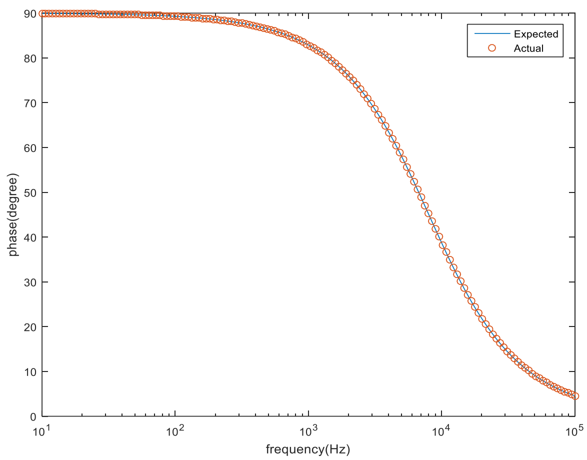

Figure 5: Phase of the frequency response of a high-pass RC circuit

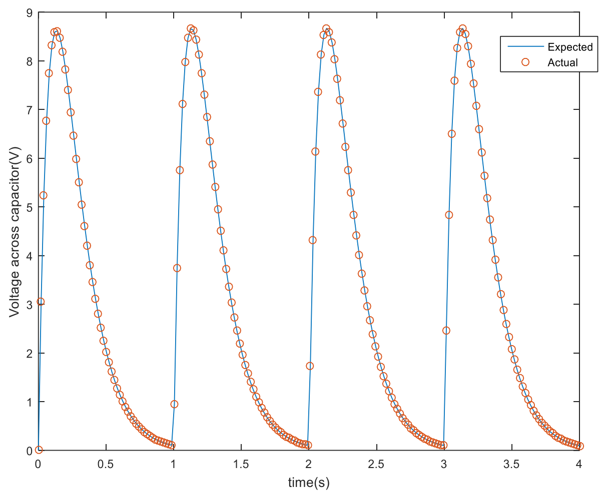

Figure 6: Behaviour of an RLC circuit – Transient response

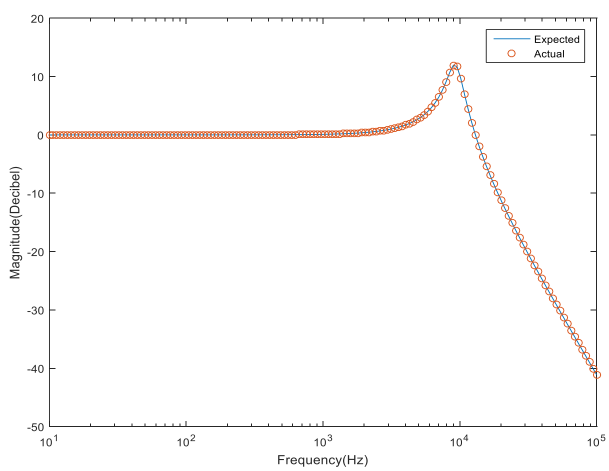

Figure 7: Amplitude of the frequency response of a low-pass RLC circuit

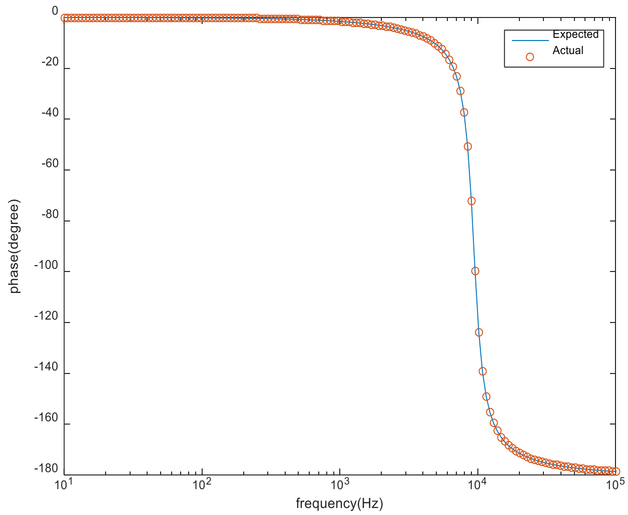

Figure 8: Phase of the frequency response of a low-pass RLC circuit

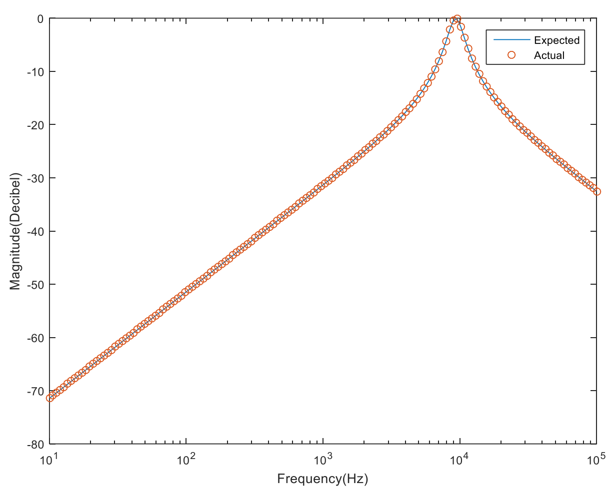

Figure 9: Amplitude of the frequency response of a high-pass RLC circuit

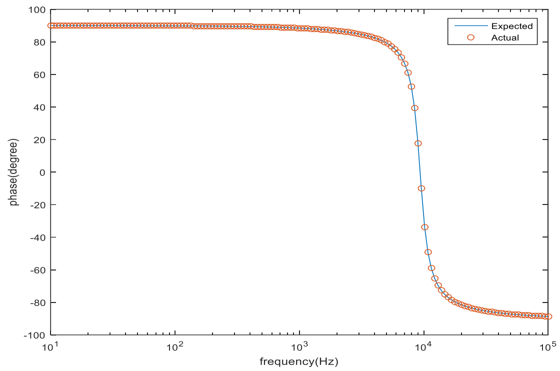

Figure 10: Phase of the frequency response of a high-pass RLC circuit

6.0 Comments and Conclusion

The transient and frequency response of RC and RLC circuits to Instantaneous forcing functions has been investigated in this study. The governing equations of electrical vibration for both circuits were formulated using Kirchhoff's voltage law. The Laplace transform method was employed to find the transfer functions of both circuits giving solutions as frequency response and transient response respectively. Each analytical solution was tested empirically against the actual behaviour of RC and RLC circuits. Results show that;

For the RC circuit,

i) The highest voltage over the time range (0, 6) occurs at $t = 1$ after which the voltage tends to zero.

ii) For frequencies higher than $2000Hz$, the low-pass RC circuit would cease to function.

iii) For the high-pass RC circuit to operate normally, frequencies higher than $8000Hz$ are required.

While for the RLC circuit,

i) A voltage surge occurs immediately after $t = 0$ and this phenomenon repeats after every second.

ii) Frequencies lower than $4 \times 10^4 Hz$ are required for the low-pas RLC circuit to function optimally.

iii) The high-pass circuit requires higher frequency than $10^4 Hz$ for it to work optimally.

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