

DYNAMIC RESPONSE OF A RAYLEIGH BEAM WITH TIME-DEPENDENT STIFFNESS UNDER MOVING LOADS WITH VARIABLE VELOCITY

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Abstract

This study examines the dynamic response of uniform beam under the action of partially distributed moving masses with variable velocity. Utilizing an analytical approach, the research investigates the impact of variable foundation stiffness, velocity variation, and structural parameters on dynamic behaviour. The results reveal significant effects of variable foundation stiffness and velocity variation on structural vibration and stability. These findings contribute to the development of more accurate design guidelines and effective vibration control strategies for elastic structures under dynamic loads.

Keywords: Elastic structures, Vibration analysis, Variable velocity, Partially distributed moving masses, Dynamic response.

1.0 INTRODUCTION

The dynamic response of elastic structures under partially distributed masses moving at varying velocities is a complex problem with significant implications for engineering design and safety (Ogunbamike, 2012; Ali & Hawwa, 2023). In literature, this behavior of elastic structures subjected to moving loads has almost exclusively been investigated under the assumption of constant load velocities (Fryba, 1972; Oni & Ogunbamike, 2010; Hsu, 2020; Yalin *et al.*, 2020; Vaccaro *et al.*, 2021; Sharma 2021; Heshmat & Elshabraway, 2021; Musa *et al.*, 2022; Ogunbamike & Owolanke, 2022; Zhao *et al.*, 2023; Baddyo *et al.*, 2024; Bao *et*

al., 2024; Liu, 2024; Kanwal *et al.*, 2024; Andi *et al.*, 2024; Jimoh *et al.*, 2025). Despite the prevalence of moving loads with variable velocities in the real world applications, little attention has been received in literature (Oni & Ogunbamike, 2018; Pireda *et al.*, 2021; Koc 2021; Can *et al.*, 2022; Deng *et al.*, 2022; Sobhanirad & Hassani, 2023). This may be as a result of non-linearity effects, complicating analysis and the intricate space-time dependencies involved in such problems. In particular, even when the inertial effects of the moving load are disregarded, analytical solutions involving integral transforms remain

intractable and cumbersome Moradi *et al.*, (2021). Several practical concerns such as vehicle acceleration and braking, time dependent factors, terrain irregularities like uneven surfaces, bumps or inclines, taking off and landing of air-crafts on runway where movement varies as a function of time, has heightened the need to study the behavior of structures subjected to loads moving with variable velocities. This class of problem was first tackled by Lowan (1935) who solved the problem of transverse oscillations in beams subjected to moving loads with variable velocity. Much later, Andi *et al.*, (2014) treated the dynamic effects on the transverse motion of a simply supported uniform beam of a load moving at variable speed. The work of Jimoh *et al.*, (2018) represents a recent advancement in this field of study. Specifically, they undertook the analysis of response of non-uniformly prestressed Timoshenko beam under distributed moving load moving at variable velocity. It was found that the moving distributed force is not an upper bound for the accurate solution of the moving distributed mass problem which shows that the inertia term must be considered for accurate assessment of the response to moving distributed load. Years later, Challah *et al.*, (2020) analyzed the vibration of continuous beams finite element method based upon the Euler-Bernoulli assumptions. They calculated the transverse vibration period for each position of the intermediate support for different end constraints. While, Usman *et al.*, (2020) considered the vibration of beam subjected to moving force and moving mass. Jiang *et al.*, (2021) investigated the dynamic responses of railway track multi-layer beam structure system under a moving load, which is connected by Winkler springs. In their work, they used modal superposition method to obtain the displacement formulas for both forced and free vibration stages of the finite

beam. Ogunbamike (2021) studied the dynamic analysis of a clamped-clamped beam under moving distributed load. The solution technique is based on the generalized finite integral transformation and a modification of the Struble's asymptotic technique. Analytical solutions showed that higher values of axial force, damping due to strain resistance and rotary inertia reduce the response amplitudes of the beam. Recently, Santos (2024) used finite element formulation to solve the dynamic analysis of Euler-Bernoulli beam subjected to moving loads. The nonlinear deformation and vibrations of beams using the Extended Rayleigh-Ritz method (ERRM) was studied by Jiang *et al.*, (2022). The dynamic response of beam supported by finite-thickness elastic under the moving load was studied by Ma *et al.*, (2024). In a more recent development, Ajijola (2025) examines the transverse displacement and rotation of a prestressed damped shear beam supported by a Vlasov foundation when subjected to a moving load travelling at a constant velocity. However, in all the aforementioned authors, the interplay between velocity variation and variable foundation stiffness which can lead to complex dynamic behavior and affect the structural stability were neglected. In this paper, apart from presenting the effect of variable velocity which can cause dynamic amplification, leading to increased stresses and deflections, the main objective of the present study is to analyze the effect of the time-dependent stiffness of the elastic structure which changes the natural frequency and affecting its dynamic response of the structure.

2.0 PROBLEM FORMULATION

The governing equation for the uniform Rayleigh beam on variable elastic foundation under an arbitrary partially distributed moving load is considered. The differential equation of the motion is given by

$$EI \frac{\partial^4 V(x,t)}{\partial x^4} - N \frac{\partial^2 V(x,t)}{\partial x^2} + \mu \frac{\partial^2 V(x,t)}{\partial t^2} - \mu R^0 \frac{\partial^4 V(x,t)}{\partial x^2 \partial t^2} + Z_F V(x,t) = P(x,t) \quad (1)$$

in which EI , N , μ , V , R^0 and Z_F are, flexural rigidity, axial force of the beam, mass per unit length, beam deflection, rotatory inertia factor and variable elastic foundation respectively. Moreover, P represents the magnitude of the distributed load given by

$$P(x,t) = P_f(x,t) \left[1 - \frac{1}{g} \frac{d^2 V(x,t)}{dt^2} \right] \quad (2)$$

$$P_f(x,t) = MgH[x - f(t)] \quad (3)$$

and

$$\frac{d^2}{dt^2} = \frac{\partial^2}{\partial t^2} + \frac{2d}{dt} f(t) \frac{\partial^2}{\partial x \partial t} + \left(\frac{df(t)}{dt} \right)^2 \frac{\partial^2}{\partial x^2} + \frac{d^2}{dt^2} f(t) \frac{\partial}{\partial x} \quad (4)$$

in which, P_f , g and $\frac{d^2}{dt^2}$ are the continuous moving force acting on the beam model, acceleration due to gravity and the convective acceleration operator. The boundary conditions are given as

$$V(0,t) = V(L,t) = 0; \quad \frac{\partial^2 V(0,t)}{\partial x^2} = \frac{\partial^2 V(L,t)}{\partial x^2} = 0 \quad (5)$$

The distance covered by the load on the same structure at any given instance of time is given as

$$f(t) = x_0 + ct + \frac{1}{2}at^2 \quad (6)$$

where x_0 is the point of application of force $P(x,t)$ at the instance $t = 0$, c is the initial velocity, a is the constant acceleration of motion and $H[x - f(t)]$ is the Heaviside function.

substituting Eqs. (2) and (4) into Eq. (1) and taken into consideration Eq. (6) to obtain

$$\begin{aligned} EI \frac{\partial^4 V(x,t)}{\partial x^4} - N \frac{\partial^2 V(x,t)}{\partial x^2} + \mu \frac{\partial^2 V(x,t)}{\partial t^2} - \mu R^0 \frac{\partial^4 V(x,t)}{\partial x^2 \partial t^2} + K(4x - 3x^2 + x^3)V(x,t) \\ + MH \left[x - \left(x_0 + ct + \frac{1}{2}at^2 \right) \right] \left[\frac{\partial^2 V(x,t)}{\partial t^2} + 2(c + at) \frac{\partial^2 V(x,t)}{\partial x \partial t} + (c + at)^2 \frac{\partial^2 V(x,t)}{\partial x^2} \right. \\ \left. + a \frac{\partial V(x,t)}{\partial x} \right] = MgH \left[x - \left(x_0 + ct + \frac{1}{2}at^2 \right) \right] \end{aligned} \quad (7)$$

where the variable elastic foundation $Z_F(x,t) = K(4x - 3x^2 + x^3)$

3.0 METHOD OF SOLUTION

The dynamic behaviour of the beam due to a partially distribute moving load is computed by using the property of Heaviside function to express it in series form and then adopt the Fourier sine integral transform which is defined as

$$V(m, t) = \int_0^L V(x, t) \sin \frac{m\pi x}{L} dx \quad (8)$$

with inverse

$$V(x, t) = \frac{2}{L} \sum_{m=1}^{\infty} V(m, t) \sin \frac{m\pi x}{L} \quad (9)$$

After implementing the Fourier transform on Eq. (7) and imposing the boundary conditions indicated in Eq. (5), one obtains

$$\begin{aligned} EI\mathfrak{R}_1 V(m, t) + V_u(m, t) - N\mathfrak{R}_2 V(m, t) + \mu R^0 \mathfrak{R}_2 V_u(m, t) + K(4x + 3x^2 + x^3)V(m, t) + \mathfrak{R}_3(t) \\ + 2(c + at)\mathfrak{R}_4(t) + (c + at)^2 \mathfrak{R}_5(t) + a\mathfrak{R}_6(t) = Mg \int_0^L H\left[x - \left(x_0 + ct + \frac{1}{2}at^2\right)\right] \sin \frac{m\pi x}{L} dx \end{aligned} \quad (10)$$

where

$$\begin{aligned} \mathfrak{R}_1 &= \frac{m^4 \pi^4}{L^4} \int_0^L \sin \frac{m\pi x}{L} dx \\ \mathfrak{R}_2 &= \frac{m^2 \pi^2}{L^2} \int_0^L \sin \frac{m\pi x}{L} dx \\ \mathfrak{R}_3 &= M \int_0^L H\left[x - \left(x_0 + ct + \frac{1}{2}at^2\right)\right] \sin \frac{m\pi x}{L} dx \\ \mathfrak{R}_4 &= M \int_0^L H\left[x - \left(x_0 + ct + \frac{1}{2}at^2\right)\right] \frac{\partial^2 V(x, t)}{\partial x \partial t} \sin \frac{m\pi x}{L} dx \\ \mathfrak{R}_5 &= M \int_0^L H\left[x - \left(x_0 + ct + \frac{1}{2}at^2\right)\right] \frac{\partial^2 V(x, t)}{\partial x^2} \sin \frac{m\pi x}{L} dx \\ \mathfrak{R}_6 &= M \int_0^L H\left[x - \left(x_0 + ct + \frac{1}{2}at^2\right)\right] \frac{\partial V(x, t)}{\partial x} \sin \frac{m\pi x}{L} dx \end{aligned} \quad (11)$$

The above integrals cannot be easily computed due to the non-differentiability of the Heaviside function at $x = 0$ and the singularity involved. In order to handle the singularities in the integrals, we use the Fourier series representation for the Heaviside unit step function

$$H\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]=\frac{1}{4}+\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)\pi\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]}{2n+1}, \quad 0 < x < L \quad (12)$$

Simplifying integral in Eq. (11) in conjunction with (12), we have

$$\begin{aligned} & \left(R^0 \frac{m^2 \pi^2}{L^2} + 1\right) V_{tt}(m, t) + \frac{1}{\mu} \left[\frac{m^2 \pi^2}{L^2} \left(N + EI \frac{m^2 \pi^2}{L^2} \right) + K(4x - 3x^2 + x^3) \right] V(m, t) + \chi_0 L \left[\frac{1}{4} + \frac{L}{\pi^2} \sum_{n=0}^{\infty} (2n+1) \right. \\ & \left. \left(\frac{(-1)^{2m} \cos(2n+1)\pi L - 1}{[(2n+1)L]^2} - \frac{\cos(2n+1)\pi L}{[(2n+1)L]^2 - 4m^2} \right) \frac{\cos(2n+1)\pi(x_0 + ct + \frac{1}{2}at^2)}{2n+1} \right] V_{tt}(m, t) - \frac{c^2 m^2 \pi^2}{L^2} \left[\frac{L}{8} + \right. \\ & \left. \frac{L^2}{2\pi^2} \sum_{n=0}^{\infty} (2n+1) \left(\frac{(-1)^{2m} \cos(2n+1)\pi L - 1}{[(2n+1)L]^2 - 4m^2} - \frac{\cos(2n+1)\pi L}{[(2n+1)L]^2} \right) \frac{\cos(2n+1)\pi(x_0 + ct + \frac{1}{2}at^2)}{2n+1} \right] V(m, t) + \\ & 2\chi_0 \left\{ \left[\frac{L^2}{2\pi^2} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} (2n+1) \left(\frac{(-1)^{p+m} \cos(2n+1)\pi L - 1}{[(2n+1)L]^2 - [(p+m)]^2} - \frac{\cos(2n+1)\pi L}{[(2n+1)L]^2} \right) \frac{\cos(2n+1)\pi(x_0 + ct + \frac{1}{2}at^2)}{2n+1} \right] V_{tt}(p, t) + \right. \\ & \left. \left[-\frac{c^2 p^2 \pi}{16} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} (2n+1) \left(\frac{(-1)^{p+m} \cos(2n+1)\pi L - 1}{[(2n+1)L]^2 - [(p+m)]^2} - \frac{\cos(2n+1)\pi L}{[(2n+1)L]^2} \right) \frac{\cos(2n+1)\pi(x_0 + ct + \frac{1}{2}at^2)}{2n+1} \right] V(p, t) \right\} \\ & = \frac{MgL}{m\pi\mu} \left[\cos \frac{m\pi}{L} (x_0 + ct + \frac{1}{2}at^2) - \cos m\pi \right] \end{aligned} \quad (13)$$

Equation (13) is the fundamental equation of our problem when the beam is resting on variable foundation stiffness under travelling distributed loads. In what follows, two special cases of (13) are discussed.

3.1 Closed Form Solution

3.1.1. Case I. The differential equation describing the flexural vibration of a finite uniform beam subjected to a moving partially distributed force may be obtained from (13) by setting $\chi_0 = 0$. In this case, we obtain

$$\begin{aligned} & \left(R^0 \frac{m^2 \pi^2}{L^2} + 1\right) V_{tt}(m, t) + \frac{1}{\mu} \left[\frac{m^2 \pi^2}{L^2} \left(N + EI \frac{m^2 \pi^2}{L^2} \right) + K(4x - 3x^2 + x^3) \right] V(m, t) \\ & = \frac{MgL}{m\pi\mu} \left[\cos \frac{m\pi}{L} (x_0 + ct + \frac{1}{2}at^2) - \cos m\pi \right] \end{aligned} \quad (14)$$

This is an approximate model which assumes the inertia effect of the moving partially distributed mass as negligible. Thus, equation (14) after some rearrangements, can be written as

$$V_{tt}(m, t) + \gamma_{jkf}^2 V(m, t) = A_{xy} \left[\cos \frac{m\pi}{L} (x_0 + ct + \frac{1}{2}at^2) - (-1)^m \right] \quad (15)$$

where

$$\gamma_{kf}^2 = \frac{\frac{1}{\mu} \left[\frac{m^2 \pi^2}{L^2} \left(N + EI \frac{m^2 \pi^2}{L^2} \right) + K(4x - 3x^2 + x^3) \right]}{R^0 \frac{m^2 \pi^2}{L^2} + 1} \quad (16)$$

$$A_{xy} = \frac{\frac{MgL}{m\pi\mu}}{R^0 \frac{m^2 \pi^2}{L^2} + 1} \quad (17)$$

Solving Eq. (15) in conjunction with the initial condition, the solution is given by

$$\begin{aligned} \bar{V}(x, t) = \frac{2}{L} \sum_{m=1}^{\infty} \frac{A_{xy}}{\gamma_{kf}} \left\{ \sin \gamma_{kf} t \left[\cos \left(\frac{b_2^2}{4a} - c_0 \right) C \left(\frac{b_2 + 2at}{\sqrt{2\pi a}} \right) + \sin \left(\frac{b_2^2}{4a} - c_0 \right) S \left(\frac{b_2 + 2at}{\sqrt{2\pi a}} \right) + \sin \left(\frac{b_1^2}{4a} - c_0 \right) S \left(\frac{b_1 + 2at}{\sqrt{2\pi a}} \right) \right. \right. \\ \left. \left. + \cos \left(\frac{b_1^2}{4a} - c_0 \right) C \left(\frac{b_1 + 2at}{\sqrt{2\pi a}} \right) - \sin \left(\frac{b_2^2}{4a} - c_0 \right) S \left(\frac{b_2}{\sqrt{2\pi a}} \right) - \cos \left(\frac{b_2^2}{4a} - c_0 \right) C \left(\frac{b_2}{\sqrt{2\pi a}} \right) - \cos \left(\frac{b_1^2}{4a} - c_0 \right) C \left(\frac{b_1}{\sqrt{2\pi a}} \right) \right. \right. \\ \left. \left. - \sin \left(\frac{b_1^2}{4a} - c_0 \right) S \left(\frac{b_1}{\sqrt{2\pi a}} \right) - \frac{1}{2\gamma_{kf}} \left[\sin(m\pi + \gamma_{kf} t) - \sin(m\pi - \gamma_{kf} t) \right] - \frac{\cos \gamma_{kf} t}{\gamma_{kf}} \left[-\sin \left(\frac{b_1^2}{4a} - c_0 \right) C \left(\frac{b_1 + 2at}{\sqrt{2\pi a}} \right) \right. \right. \right. \\ \left. \left. + \cos \left(\frac{b_2^2}{4a} - c_0 \right) S \left(\frac{b_2 + 2at}{\sqrt{2\pi a}} \right) + \sin \left(\frac{b_2^2}{4a} - c_0 \right) C \left(\frac{b_2 + 2at}{\sqrt{2\pi a}} \right) - \cos \left(\frac{b_2^2}{4a} - c_0 \right) S \left(\frac{b_2 + 2at}{\sqrt{2\pi a}} \right) + \sin \left(\frac{b_1^2}{4a} - c_0 \right) C \left(\frac{b_1}{\sqrt{2\pi a}} \right) \right. \right. \\ \left. \left. - \cos \left(\frac{b_1^2}{4a} - c_0 \right) S \left(\frac{b_1}{\sqrt{2\pi a}} \right) - \sin \left(\frac{b_2^2}{4a} - c_0 \right) C \left(\frac{b_2}{\sqrt{2\pi a}} \right) + \cos \left(\frac{b_2^2}{4a} - c_0 \right) S \left(\frac{b_2}{\sqrt{2\pi a}} \right) + \frac{\cos m\pi}{\gamma_{kf}} \right. \right. \\ \left. \left. + \frac{1}{2\gamma_{kf}} \left[\cos(m\pi - \gamma_{kf} t) + \cos(m\pi + \gamma_{kf} t) \right] \right] \right\} \left(\sin \frac{m\pi}{L} x \right) \quad (18) \end{aligned}$$

Eq. (18) represents the transverse displacement response to a partially distributed force moving at variable velocities of a beam with a variable foundation constant.

3.1.2. Case II. If the inertia term is retained, then $\chi_0 \neq 0$. This is termed as moving mass problem. In this case the solution of the entire equation (13) is required. The problem does not have an exact solution, hence using a modification of Struble technique discussed in Oni & Ogunbamike (2018), equation (13) is simplified and rearranged to take the form

$$\begin{aligned}
& V_{tt}(m, t) + \gamma_{kf}^2 V(m, t) + \left[\frac{\chi_0 L}{R^0 \frac{m^2 \pi^2}{L^2} + 1} \right] \left[\frac{1}{4} + \frac{L}{\pi^2} \sum_{n=0}^{\infty} (2n+1) \left(\frac{(-1)^{2m} \cos(2n+1)\pi L - 1}{[(2n+1)L]^2} - \right. \right. \\
& \left. \left. \frac{\cos(2n+1)\pi L}{[(2n+1)L]^2 - 4m^2} \right) \frac{\cos(2n+1)\pi(x_0 + ct + \frac{1}{2}at^2)}{2n+1} \right] V_{tt}(m, t) - \frac{c^2 m^2 \pi^2}{L^2} \left[\frac{L}{8} + \frac{L^2}{2\pi^2} \sum_{n=0}^{\infty} (2n+1) \right. \\
& \left. \left(\frac{(-1)^{2m} \cos(2n+1)\pi L - 1}{[(2n+1)L]^2 - 4m^2} - \frac{\cos(2n+1)\pi L}{[(2n+1)L]^2} \right) \frac{\cos(2n+1)\pi(x_0 + ct + \frac{1}{2}at^2)}{2n+1} \right] V(m, t) + 2\chi_0 \left\{ \left[\frac{L^2}{2\pi^2} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} (2n+1) \right. \right. \\
& \left. \left(\frac{(-1)^{p+m} \cos(2n+1)\pi L - 1}{[(2n+1)L]^2 - [(p+m)]^2} - \frac{\cos(2n+1)\pi L}{[(2n+1)L]^2} \right) \frac{\cos(2n+1)\pi(x_0 + ct + \frac{1}{2}at^2)}{2n+1} \right] V_{tt}(p, t) + \\
& \left. \left[-\frac{c^2 p^2 \pi}{16} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} (2n+1) \left(\frac{(-1)^{p+m} \cos(2n+1)\pi L - 1}{[(2n+1)L]^2 - [(p+m)]^2} - \frac{\cos(2n+1)\pi L}{[(2n+1)L]^2} \right) \frac{\cos(2n+1)\pi(x_0 + ct + \frac{1}{2}at^2)}{2n+1} \right] V(p, t) \right\} \\
& = \frac{MgL}{m\pi\mu \left[R^0 \frac{m^2 \pi^2}{L^2} + 1 \right]} \left[\cos \frac{m\pi}{L} (x_0 + ct + \frac{1}{2}at^2) - (-1)^m \right] \quad (19)
\end{aligned}$$

Firstly, we shall consider the homogeneous part of equation (19) and obtain a modified frequency corresponding to the frequency of the free system. An equivalent free system operator defined by the modified frequency then replaces equation (19). Considering a parameter $\xi_0 < 1$ for any arbitrary mass ratio χ_0 defined as

$$\xi_0 = \frac{\chi_0}{1 + \chi_0} \quad (20)$$

It is then clear that

$$\chi_0 = \xi_0 [1 + o(\xi_0) + o(\xi_0^2) + \dots] \quad (21)$$

When ξ_0 is set to zero in equation (19), a case corresponding to the case when the inertia effect of the mass of the system is neglected and the solution of (19) is given as

$$\bar{V}(m, t) = \Psi_0 \cos[\gamma_{kf} t - \tilde{\lambda}_m] \quad (22)$$

Ψ_0 and $\tilde{\lambda}_m$ are constants. Since $\xi_0 \ll 1$, Struble's technique requires that the asymptotic solution of the homogeneous part of equation (19) which can be written as

$$\bar{V}(m, t) = \Psi(m, t) \cos[\gamma_{kf} t - \tilde{\lambda}(m, t)] + \xi_0 \bar{V}(m, t) + o(\xi_0) \quad (23)$$

where $\Psi(m, t)$ and $\tilde{\lambda}(m, t)$ are slowly varying functions of time.

Substituting equation (23) and its derivatives into the homogeneous part of (19) and taking into account (21) while retaining terms to $O(\xi_0)$, one obtains

$$\begin{aligned}
& 2\Psi(m,t)\dot{\lambda}(m,t)\cos[\gamma_{kf}t-\dot{\lambda}(m,t)]-2\dot{\Psi}(m,t)\gamma_{kf}\sin[\gamma_{kf}t-\dot{\lambda}(m,t)]+\Psi(m,t)\gamma_{kf}^2\cos[\gamma_{kf}t-\dot{\lambda}(m,t)] \\
& +\left[\frac{\xi_0 L}{R^0\frac{m^2\pi^2}{L^2}+1}\right]\left[\frac{1}{4}+\frac{L}{\pi^2}\sum_{n=0}^{\infty}(2n+1)\left(\frac{(-1)^{2m}\cos(2n+1)\pi L-1}{[(2n+1)L]^2}-\frac{\cos(2n+1)\pi L}{[(2n+1)L]^2-4m^2}\right)\right. \\
& \left.\frac{\cos(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}\right]\left[2\Psi(m,t)\dot{\lambda}(m,t)\cos[\gamma_{kf}t-\dot{\lambda}(m,t)]-2\dot{\Psi}(m,t)\gamma_{kf}\sin[\gamma_{kf}t-\dot{\lambda}(m,t)]\right] \\
& -\frac{c^2m^2\pi^2}{L^2}\left[\frac{L}{8}+\frac{L^2}{2\pi^2}\sum_{n=0}^{\infty}(2n+1)\left(\frac{(-1)^{2m}\cos(2n+1)\pi L-1}{[(2n+1)L]^2-4m^2}-\frac{\cos(2n+1)\pi L}{[(2n+1)L]^2}\right)+\right. \\
& \left.\frac{\cos(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}\right]\Psi(m,t)\gamma_{kf}^2\cos[\gamma_{kf}t-\dot{\lambda}(m,t)]+2\xi_0\left\{\left[\frac{L^2}{2\pi^2}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}(2n+1)\right.\right. \\
& \left.\left.\left(\frac{(-1)^{p+m}\cos(2n+1)\pi L-1}{[(2n+1)L]^2-[(p+m)]^2}-\frac{\cos(2n+1)\pi L}{[(2n+1)L]^2}\right)\frac{\cos(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}\right]\right. \\
& \left.2\Psi(p,t)\dot{\lambda}(p,t)\cos[\gamma_{kf}t-\dot{\lambda}(p,t)]-2\dot{\Psi}(p,t)\gamma_{kf}\sin[\gamma_{kf}t-\dot{\lambda}(p,t)]+\left[-\frac{c^2p^2\pi}{16}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}(2n+1)\right.\right. \\
& \left.\left.\left(\frac{(-1)^{p+m}\cos(2n+1)\pi L-1}{[(2n+1)L]^2-[(p+m)]^2}-\frac{\cos(2n+1)\pi L}{[(2n+1)L]^2}\right)\frac{\cos(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}\right]\right. \\
& \left.\Psi(p,t)\gamma_{kf}^2\cos[\gamma_{kf}t-\dot{\lambda}(p,t)]\right]=\frac{MgL}{m\pi\mu\left[R^0\frac{m^2\pi^2}{L^2}+1\right]}\left[\cos\frac{m\pi}{L}(x_0+ct+\frac{1}{2}at^2)-(-1)^m\right] \quad (24)
\end{aligned}$$

The variational part of the equation describing the behaviour of $\Psi(m,t)$ and $\dot{\lambda}(m,t)$ during the motion of the distributed mass is extracted. Neglecting terms that do not contribute to the variational equation and taking into account the trigonometric identities, Eq. (24) reduces to

$$\begin{aligned}
& 2\Psi(m,t)\dot{\lambda}(m,t)\cos[\gamma_{kf}t-\dot{\lambda}(m,t)]-2\dot{\Psi}(m,t)\gamma_{kf}\sin[\gamma_{kf}t-\dot{\lambda}(m,t)]+\Psi(m,t)\gamma_{kf}^2\cos[\gamma_{kf}t-\dot{\lambda}(m,t)] \\
& +\frac{\xi_0 L}{4\Delta_1}\left[-\frac{c^2m^2\pi^2}{8L}\Psi(m,t)\gamma_{kf}^2\cos[\gamma_{kf}t-\dot{\lambda}(m,t)]\right]=0 \quad (25)
\end{aligned}$$

The variational equations are obtained by setting the coefficients of $\cos[\gamma_{kf}t-\dot{\lambda}(m,t)]$ and $\sin[\gamma_{kf}t-\dot{\lambda}(m,t)]$ in Eq. (25) to zero. Thus

$$-2\dot{\Psi}(m,t)\gamma_{kf}=0 \quad (26)$$

and

$$2\Psi(m,t)\dot{\lambda}(m,t)+\frac{\xi_0 L}{4\Delta_1}\left[2\Psi(m,t)-\frac{c^2m^2\pi^2\Psi(m,t)\gamma_{kf}^2}{8L}\right]=0 \quad (27)$$

Solving equations (26) and (27) gives



$$\Psi(m, t) = H_m \quad (28)$$

and

$$\lambda(m, t) = \frac{\xi_0 L}{2\Delta_1} \left(1 - \frac{c^2 m^2 \pi^2 \gamma_{kf}^2}{16L} \right) t + K_m \quad (29)$$

respectively, where H_m and K_m are constants.

The first approximation to the homogeneous system when the mass effect is considered is

$$\bar{V}(m, t) = \Im_m \cos[\beta_{mm} t - K_m] \quad (30)$$

where

$$\beta_{mm} = \gamma_{kf} \left\{ 1 - \frac{\xi_0}{16} \left(\frac{8L^2 - c^2 m^2 \pi^2 \gamma_{kf}^2}{\Delta_1 L} \right) \right\} \quad (31)$$

Hence the entire Eq. (19) can be written in the form

$$\bar{V}_u(m, t) + \beta_{mm}^2 \bar{V}(m, t) = \frac{\xi_1 L^2 g}{m\pi\Delta_1} \left[\cos \frac{m\pi}{L} (x_0 + ct + \frac{1}{2} at^2) - (-1)^m \right] \quad (32)$$

Solving equation (32) using method of variation of parameters technique in conjunction with the initial condition, one obtains expression for $\bar{V}(m, t)$

$$\begin{aligned} \bar{V}(x, t) = & \frac{2}{L} \sum_{m=1}^{\infty} \frac{\xi_0 L^2 g}{\beta_{mm} \Delta_1} \left\{ \sin \beta_{mm} t \left[\cos \left(\frac{b_2^2}{4a} - c_0 \right) C \left(\frac{b_2 + 2at}{\sqrt{2\pi a}} \right) + \sin \left(\frac{b_2^2}{4a} - c_0 \right) S \left(\frac{b_2 + 2at}{\sqrt{2\pi a}} \right) + \sin \left(\frac{b_1^2}{4a} - c_0 \right) S \left(\frac{b_1 + 2at}{\sqrt{2\pi a}} \right) \right. \right. \\ & + \cos \left(\frac{b_1^2}{4a} - c_0 \right) C \left(\frac{b_1 + 2at}{\sqrt{2\pi a}} \right) - \sin \left(\frac{b_2^2}{4a} - c_0 \right) S \left(\frac{b_2}{\sqrt{2\pi a}} \right) - \cos \left(\frac{b_2^2}{4a} - c_0 \right) C \left(\frac{b_2}{\sqrt{2\pi a}} \right) - \cos \left(\frac{b_1^2}{4a} - c_0 \right) C \left(\frac{b_1}{\sqrt{2\pi a}} \right) \\ & - \sin \left(\frac{b_1^2}{4a} - c_0 \right) S \left(\frac{b_1}{\sqrt{2\pi a}} \right) - \frac{1}{2\beta_{mm}} \left[\sin(m\pi + \beta_{mm} t) - \sin(m\pi - \beta_{mm} t) \right] - \frac{\cos \beta_{mm} t}{\beta_{mm}} \left[-\sin \left(\frac{b_1^2}{4a} - c_0 \right) C \left(\frac{b_1 + 2at}{\sqrt{2\pi a}} \right) \right. \\ & + \cos \left(\frac{b_1^2}{4a} - c_0 \right) S \left(\frac{b_1 + 2at}{\sqrt{2\pi a}} \right) + \sin \left(\frac{b_2^2}{4a} - c_0 \right) C \left(\frac{b_2 + 2at}{\sqrt{2\pi a}} \right) - \cos \left(\frac{b_2^2}{4a} - c_0 \right) S \left(\frac{b_2 + 2at}{\sqrt{2\pi a}} \right) + \sin \left(\frac{b_2^2}{4a} - c_0 \right) C \left(\frac{b_2}{\sqrt{2\pi a}} \right) \\ & - \cos \left(\frac{b_2^2}{4a} - c_0 \right) S \left(\frac{b_2}{\sqrt{2\pi a}} \right) - \sin \left(\frac{b_1^2}{4a} - c_0 \right) C \left(\frac{b_1}{\sqrt{2\pi a}} \right) + \cos \left(\frac{b_1^2}{4a} - c_0 \right) S \left(\frac{b_1}{\sqrt{2\pi a}} \right) + \frac{\cos m\pi}{\beta_{mm}} \\ & \left. \left. + \frac{1}{2\beta_{mm}} \left[\cos(m\pi - \beta_{mm} t) + \cos(m\pi + \beta_{mm} t) \right] \right] \right\} \left(\sin \frac{m\pi}{L} x \right) \quad (33) \end{aligned}$$

which represents the dynamic response to a partially distributed mass moving with non-uniform velocity of a simply supported beam on a variable foundation constant.

4.0 NUMERICAL RESULTS AND DISCUSSION

In order to illustrate the preceding analysis, an elastic beam of length $150m$ is considered. The mass per unit length of the beam $\mu = 2758.291kg/m$, the modulus of elasticity $E = 3.1 \times 10^{10} N/m$ and $I = 2.87698 \times 10^{-3} m^4$.

A partially distributed load travels along the beam with velocity $v = 8.128m^{-1}$, the values of the foundation modulus is varied $0N/m^3$ and the values of rotatory inertia correction factor R_0 is varied from 0.5 and 5.5 .

Figure 5.1 displays the displacement response of a beam to moving partially distributed forces for various values of foundation stiffness K

and fixed values of axial force N and rotatory inertia correction factor $R^0 = 0.5$. The figure shows that as the foundation stiffness K increases, the response amplitude of the beam decreases. Similar results are obtained when the beam is subjected to moving partially distributed masses as shown in Figure 5.3. Figure 5.2 illustrates that for values of axial force, and fixed values of foundation stiffness and rotatory inertia correction factor, the deflection profile of the vibrating structure is reduced. The same results were obtained when the elastic is transverse by a partially distributed mass, as indicated in Figure 5.4. In Figure 5.5, the response of the beam to partially

distributed forces values of rotatory inertia correction factor and for fixed value of foundation stiffness and axial force. It is seen that the deflection of the beam decreases with the increase in the rotatory inertia correction factor. The same behavior characterizes the deflection profile of the uniform beam when it is traversed by moving partially distributed masses as shown in Figure 5.6. Finally, Figure 5.7 depicts the comparison of the transverse displacement response of moving partially distributed force and moving partially distributed mass of a Rayleigh beam traversed by a moving load travelling at variable velocity for fixed values of foundation stiffness, axial force and rotatory inertia correction factor.

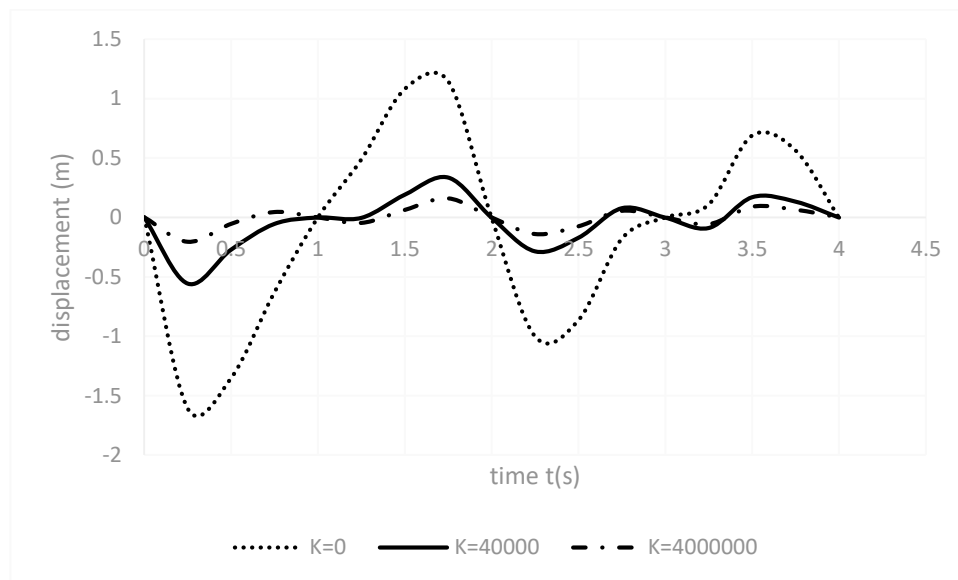


Figure 1. Deflection profile of the simply supported beam traversed by a moving partially distributed force moving at variable velocities for various K and fixed $N(20000)$, $R^0(0.5)$

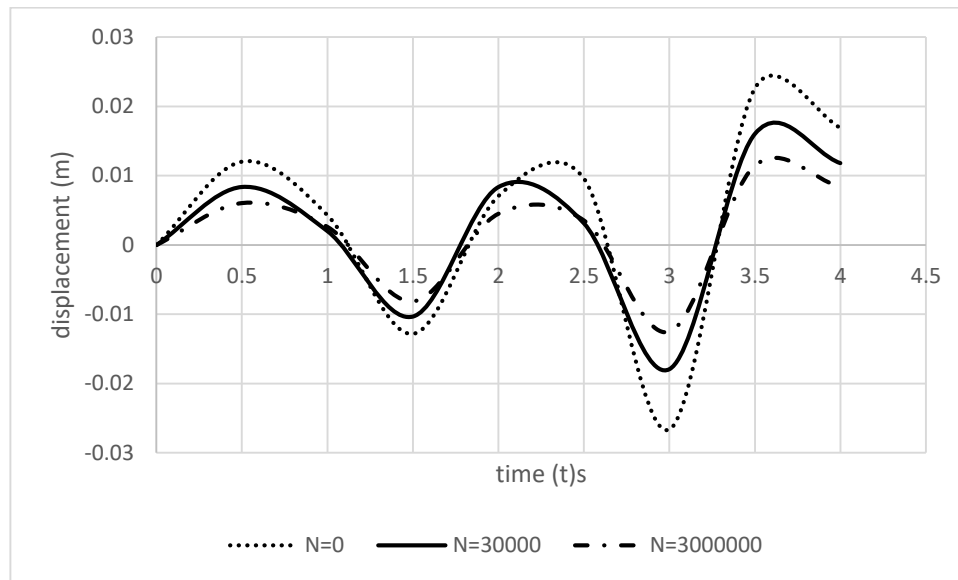


Figure 3. Transverse displacement of the simply supported beam traversed by a moving partially distributed force moving at variable velocities for various N and fixed $K(40000)$, $R^0(0.5)$

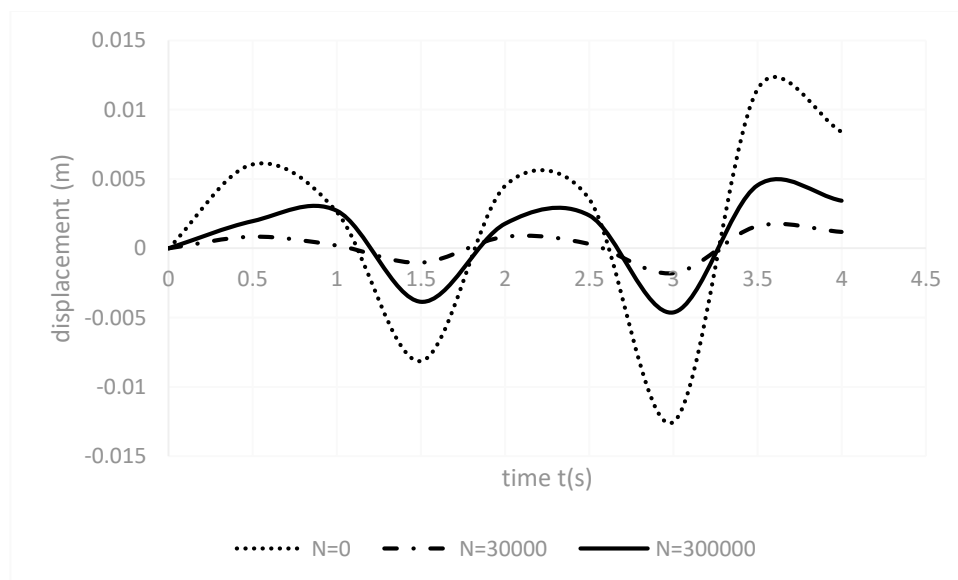


Figure 4. Transverse displacement of the simply supported beam traversed by a moving partially distributed masses moving at variable velocities for various N and fixed $K(40000)$, $R^0(0.5)$ and

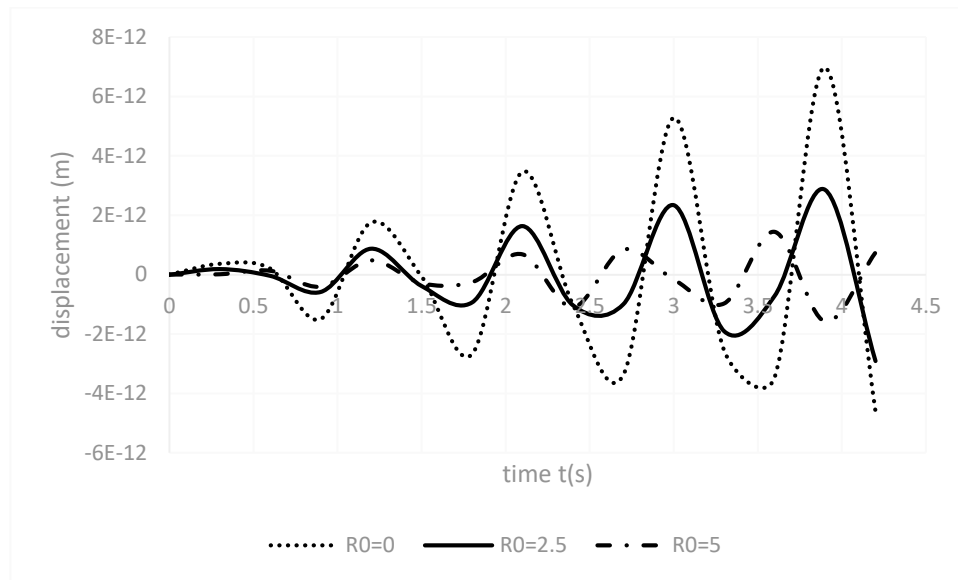


Figure 5. Response of the simply supported beam traversed by a moving partially distributed force moving at variable velocities for various R^0 and fixed $K(40000), N(20000)$

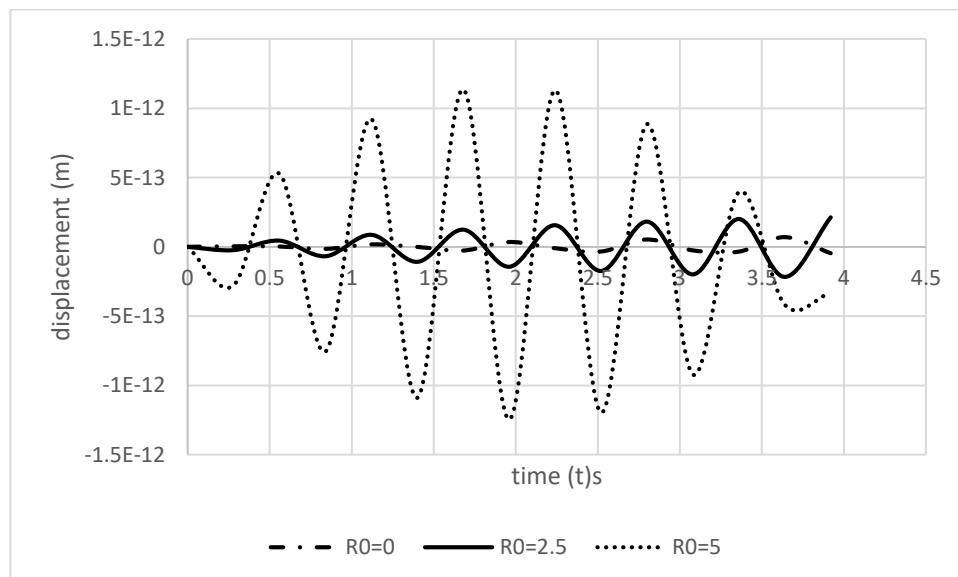


Figure 6. Response of the simply supported beam traversed by a moving partially distributed masses moving at variable velocities for various R^0 and fixed $K(40000), N(20000)$

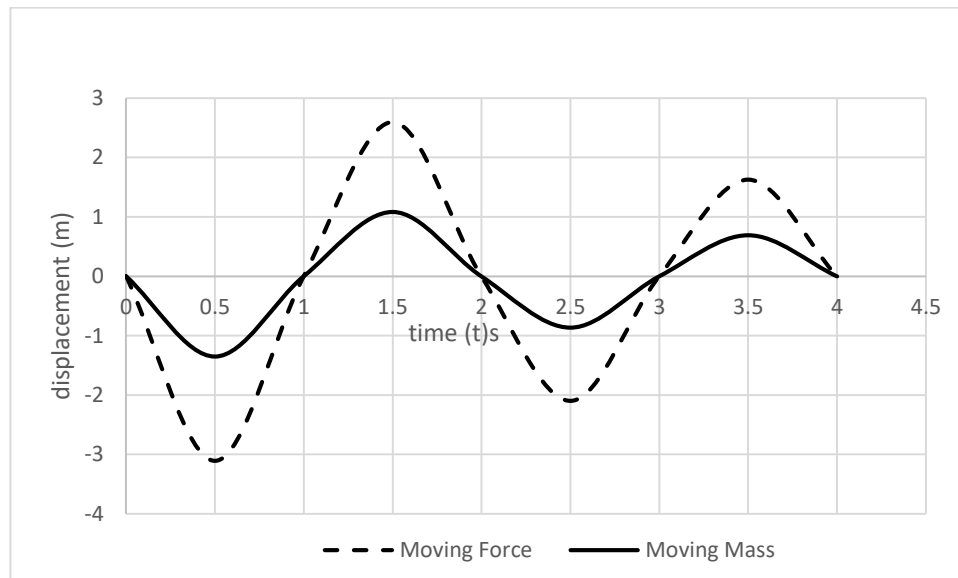


Figure 7. Comparison of the displacement response of moving partially distributed force and moving partially distributed mass moving at variable velocities for fixed values of foundation stiffness, axial force and rotatory inertia correction factor.

5.0 CONCLUSION

A close form solution is presented for the displacement response of a uniform beam under the actions of partially distributed masses moving with variable velocity. The study demonstrates the critical influence of time-dependent stiffness and velocity variation on the dynamic response of elastic structures. The analytical approach employed provides valuable insights into the complex interactions between moving masses, structural parameters, and dynamic behavior. The findings of this research have significant implications on the design and vibration control of elastic structures subjected to dynamic loads, highlighting the need for more accurate and robust design guidelines. By shedding light on the dynamic response of elastic structures, this study contributes to the advancement of structural engineering, vibration analysis, and related fields,

ultimately enhancing the safety, durability, efficiency, and reliability of engineering systems.

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Author contribution Dr Ogunbamike Oluwatoyin Kehinde designed the study, wrote the protocol and manage literature search. Dr Bagbe Atinuke and Dr. Owolanke Ayodele Olakiitan evaluate the closed form solution numerically and also proofread the work.

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