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# ON THE INVERSE EIGENVALUE PROBLEM OF A EULER BERNOULLI BEAM UNDER DISTRIBUTED LOADS

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**Abstract**

The problem of posing and solving an inverse eigenvalue problem of a Euler Bernoulli steel beam under distributed loads is studied in this paper. The governing equations are tackled using finite Fourier Sine transform for the simple support end condition and finite Fourier Cosine transform for the guided/sliding end condition. Graphs of the solution for the two end conditions considered are plotted with the first frequency $(\lambda_{n1}, \gamma_{n1})$ and imposed natural frequency $(\lambda_{n2}, \gamma_{n2})$ respectively. Deductions reveal that the sliding/guided end condition provides the best and safest vibration pattern under (imposed) natural frequencies$(\lambda_{n2}, \gamma_{n2})$ considering the regular and damped vibration as well as the minimum deflection that ensued. Conditions under which resonance occurs for the Euler Bernoulli beam are also established for the two end conditions.

**Keywords**: Inverse eigenvalue, Imposed natural frequency, Distributed loads, Euler Bernoulli beam,

## 1. INTRODUCTION

The problem of determining the natural frequency of a dynamical system has been the subject of extensive research. Dynamical systems have enormous application in many branches of Applied Mathematics, Transportation, Engineering, Physics and other related fields. This has led to an increasing need for the continuous study of the behavior of these systems subjected to moving loads.

Several researchers have made attempts at the study and they include, Davis *et al* (1972) who presented curved beam finite elements for out of plane coupled bending and torsional vibration. The element formulation was based upon the exact differential equations of an infinitesimal element in static equilibrium. The element stiffness and mass matrices were restricted to those of a thin beam without secondary effects. Frequencies obtained were shown t converge onto exact values.

Sarma and Varadam (1984) who used a Ritz finite element approach to study the large amplitude free flexural vibrations of beams with immovable ends. The formulation was based on Lagrange's equation of motion with the definition of the time function at an instant corresponding to the point of reversal. The solution for nonlinear equations was sought by using an algorithm – the direct iteration technique. The nonlinear frequencies, mode shapes for transverse and longitudinal displacements were determined for the simply supported, clamped-clamped and simply supported clamped beams. In almost all the cases, the nonlinear frequency values were found to be the lower bound.

Years later, Kukla and Posiadala (1994) investigated the lateral vibration of a loaded beam with intermediate elastic supports and concentrated masses by applying the Green functions method. The solution contained all possible combinations of classical end conditions. Numerical examples showed the influence of the attached masses on the frequency of the system. They also showed the existence of additional frequencies of the system considered compared to those of the beam without masses attached.

While, Low (1998) considered a beam with concentrated mass and performed a frequency analysis of loaded beams for ten classical beams involving guided, fixed, free and pinned ends. The explicit frequency equations were obtained by satisfying the differential equations of the eigen value problem, the boundary and compatibility equations.

Cha (2005) proposed a simple approach that can be used to readily determine the eigenvalues of an arbitrarily supported single-span/multispan beam carrying any combination of lumped mass, rotary inertia, grounded translational or torsional viscous damper. The solution to the formulated frequency equation was obtained numerically.

In another study, Gosai and Mukherjee (2010) used analytical method to derive the natural frequencies of prismatic beams subjected to various boundary conditions, and subsequently arrived at the mode shapes for the corresponding natural frequencies. The analysis was done in MATLAB. The governing equation of the tapered beam was solved using FEM and an algorithm derived to solve the frequency equations resulting from the solution to the governing differential equations of beams having different boundary conditions.

Hosseini and Baddour (2014) investigated extensively the problem of determining the eigenvalues of a vibrational system having multiple lumped attachments. A method to impose two natural frequencies on a dynamical system consisting of a Euler-Bernoulli beam and carrying a single mass attachment was evaluated.

Much later, Patil and Vibhute (2019) illustrated the comparative study of various methods of modal analysis, which included analytical method, finite element analysis and experimental method. The modal analysis was used to understand the dynamic properties of structure such as natural frequency, damping ratio and mode shape. The natural frequency of the continuous beam system was found out for different variables of beam using ANSYS 19. Findings showed that all results obtained were nearly equal. While the percentage error between the FEA approach and experimental approach lay within five percent to ten percent range.

Klanner and Ellermann (2020) solved general beam vibration problems with several attachments under arbitrarily distributed harmonic loading. A multi-spanned beam was modeled by the Euler-Bernoulli beam theory and an extension of an efficient numerical method called Numerical Assembly Technique (NAT) was used to calculate the steady-state harmonic response of the beam to an arbitrarily distributed force or moment loading. All classical boundary conditions were considered.

Gowda *et al* (2022) developed an artificial neural network and linear regression algorithm model to estimate relationship between copper materials, angular frequency and natural frequencies for free vibrations obtained by Euler Bernoulli method and ANSYS 14.5 software as an output layer. Results showed that AI can be efficiently adapted to modal analysis of beams. The graph behavior on the natural frequency from AI was also demonstrated. The beam was simply supported and cantilevered.

However, it is noted that in the aforementioned studies, researches were concerned with the forward problem of determining the eigenvalues of combined system assuming that all its physical characteristics are known. In few cases where the inverse problem is considered, the authors were concerned with imposing nodes on specific locations along the beam.

Therefore in this research work, the problem of imposing a second natural frequency on a Euler Bernoulli steel beam subject to distributed loads having simple support and guided/sliding end condition respectively is studied.

## 2. GOVERNING DIFFERENTIAL EQUATION

The governing equation for the uniform Euler Bernoulli beam carrying a distributed load of weight W and travelling at a constant speed c is given as

$$\frac{EI\partial^4 y(x,t)}{\partial x^4} + \frac{\mu\partial^2 y(x,t)}{\partial t^2} = WH(x-ct),\ 0 < x < L \qquad (1)$$

Where $\mu$ is mass per unit length of the beam, $y(x,t)$ is the deflection of the beam, $E$ is the modulus elasticity of the beam, $I$ is the area moment of inertia, $x$ is the longitudinal axis of the beam, $L$ is the length of the beam, and t is assumed to be limited to that interval of time (in seconds), when the load is on the beam $0 \leq ct \leq L$, $H(x-ct)$ is the Heaviside function used to represent the distribution load defined as:

$$H(x-ct) = \begin{cases} 0 & x < ct \\ 1 & x \geq ct \end{cases} \qquad (2)$$

For a simply supported beam, the deflections as well as the bending moments at both ends vanish. Thus, for the distributed load problems, the boundary conditions are given by:

$$y(0,t) = 0 = y(L,t) \qquad (3)$$

$$\frac{\partial^2 y(0,t)}{\partial x^2} = 0 = \frac{\partial^2 y(L,t)}{\partial x^2} \qquad (4)$$

For the sliding/guided end condition, the slope and the shear force must vanish at both ends. The boundary conditions are given by;

$$\frac{\partial y(0,t)}{\partial x} = 0 = \frac{\partial y(L,t)}{\partial x} \qquad (5)$$

$$\frac{\partial^3 y(0,t)}{\partial x^3} = 0 = \frac{\partial^3 y(L,t)}{\partial x^3} \qquad (6)$$

And the initial condition for the problem is given as:

$$y(x,0) = 0 = \frac{\partial\, y(x,0)}{\partial t} \qquad (7)$$

## 3 METHOD OF SOLUTION

### 3.1 Beam with simple supports

For solution of equation (1) subject to condition (3) and (4) the method of finite Fourier Sine transform was employed.

If $z(x)$ is a function which is sectionally continuous over the range $(0, L)$. Then the finite

Fourier Sine transform of z(x) on this interval is defined as:

$$F_s[z[x]] = z_s(n) = \int_0^L z(x)\sin\frac{n\pi x}{L}dx \quad (8)$$

where n is an integer.

The corresponding inversion formula for the finite Fourier Sine transform is given as,

$$z(x) = \frac{2}{L}\sum_1^\infty z_s(n)\sin\frac{n\pi x}{L} \quad (9)$$

Applying the finite Fourier Sine transform to each term of equation (1), an ordinary differential equation is obtained:

$$\int_0^L \frac{\partial^4 y(x,t)}{\partial x^4}\sin\frac{n\pi x}{L}dx = \frac{n^4\pi^4}{L^4}y_s(n,t) \quad (10)$$

$$\int_0^L \frac{\partial^2 y(x,t)}{\partial t^2}\sin\frac{n\pi x}{L}dx = \frac{d^2}{dt^2}y_s(n,t) \quad (11)$$

$$\int_0^L H(x-ct)\sin\frac{n\pi x}{L}dx = \frac{L}{n\pi}\left[\!\left[\cos\frac{n\pi ct}{L} - (-1)^n\right]\!\right] \quad (12)$$

In view of equations (10),(11) and (12), equation (1) now becomes,

$$EI\frac{n^4\pi^4}{L^4}y_s(n,t) + \mu\frac{d^2}{dt^2}y_s(n,t) = \frac{LW}{n\pi}\left[cos\frac{n\pi Ct}{L} - (-1)^n\right] \quad (13)$$

Equation (13) is now tackled using the method of characteristics and undetermined coefficients.

For the homogeneous part of equation (13), the complimentary function is,

$$y_{sc}(n,t) = A_1\cos\lambda_{n1}t + A_2\sin\lambda_{n1}t \quad (14)$$

where,

$$\lambda_{n1} = \pm\frac{n^2\pi^2}{L^2}\sqrt{\frac{EI}{\mu}} \quad (15)$$

is the natural frequency.

For the non- homogeneous part of equation (13), the particular integral is,

$$y_{sp}(n,t) = \frac{WL^5(-1)^{n+1}}{EIn^5\pi^5} + \frac{W}{\frac{ELn^5\pi^5}{L^5} - \frac{\mu c^2\pi^3 n^3}{L^3}}\cos\frac{n\pi ct}{L} \quad (16)$$

And the general solution $y_G(n,t)$ to equation (1) is obtained as,

$$y_G(n,t) = A_1\cos\lambda_{n1}t + A_2\sin\lambda_{n1}t + \frac{WL^5(-1)^{n-1}}{ELn^5\pi^5} + \frac{W}{\frac{EIn^5\pi^5}{L^5} - \frac{\mu c^2\pi^3 n^3}{L^3}}\cos\frac{n\pi ct}{L} \quad (17)$$

Taking the inverse finite Fourier sine transform of equation (17), one arrives at,

$$y(x,t) = \frac{2}{L}\sum_{n=1}^\infty\left(A_1\cos\lambda_{n1}t + A_2\sin\lambda_{n1}t + \frac{WL^5(-1)^{n+1}}{EIn^5\pi^5} + \frac{W}{EI\frac{n^5\pi^5}{L^5} - \mu c^2\frac{n^3\pi^3}{L^3}}\cos\frac{n\pi ct}{L}\right)\sin\frac{n\pi x}{L} \quad (18)$$

Where $A_1$ and $A_2$ are determined using the initial conditions in (7) as,

$$A_1 = -\frac{2}{L}\left(\frac{WL^5(-1)^{n+1}}{EIn^5\pi^5} + \frac{W}{EI\frac{n^5\pi^5}{L^5} - \mu c^2\frac{n^3\pi^3}{L^3}}\right) \quad (19)$$

and,

$$A_2 = 0 \quad (20)$$

With $A_1$ and $A_2$ given in (19) and (20), equation (18) becomes,

$$y(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\left(\frac{WL^5(-1)^{n+1}(1-\cos\lambda_{n1}t)}{EIn^5\pi^5} + \frac{W\left(\cos\frac{n\pi ct}{L} - \cos\lambda_{n1}t\right)}{\lambda_{n1}-\alpha_s}\right)\sin\frac{n\pi x}{L} \quad (21)$$

where,

$$\alpha_s = \frac{\mu c^4}{EI} \quad (22)$$

Equation (21) is the solution to the distributed load problem when the Euler Bernoulli beam has simple supports.

### 3.2 Beam with sliding/guided end conditions

For solution of equation (1) subject to conditions (5) and (6) the method of finite Fourier Cosine transform is employed.

If $z(x)$ is a function which is sectionally continuous over the range $(0, L)$. Then the finite Fourier Cosine transform of $z(x)$ on this interval is defined as:

$$F_c[z[x]] = z_c(n) = \int_0^L z(x)\cos\frac{n\pi x}{L}dx \quad (23)$$

where n is an integer.

The corresponding inversion formula for the finite Fourier Cosine transform is given as,

$$z(x) = \frac{2}{L}\sum_1^{\infty} z_c(n)\cos\frac{n\pi x}{L} \quad (24)$$

Applying the finite Fourier Cosine transform to each term in equation (1) gives,

$$EI\frac{n^4\pi^4}{L^4}y_c(n,t) + \mu\frac{d^2}{dt^2}y_c(n,t) = \frac{L}{n\pi}\left[\sin\frac{n\pi ct}{L}\right] \quad (25)$$

Equation (25) is also solved using the method of characteristics and undetermined coefficients with the complimentary function as,

$$y_{cc}(n,t) = D_1\cos\lambda_{n2}t + D_2\sin\lambda_{n2}t \quad (26)$$

and particular integral,

$$y_{cp}(n,t) = \frac{W}{\lambda_{n2}-\alpha_s}\sin\frac{n\pi ct}{L} \quad (27)$$

The general solution $y_G(n,t)$ to equation (25) is;

$$y_G(n,t) = D_1\cos\lambda_{n2}t + D_2\sin\lambda_{n2}t + \frac{W}{\lambda_{n2}-\alpha_s}\sin\frac{n\pi ct}{L} \quad (28)$$

Taking inverse finite Fourier Cosine transform of equation (28) yields,

$$y(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\left(D_1\cos\lambda_{n2}t + D_2\sin\lambda_{n2}t + \frac{W}{\lambda_{n2}-\alpha_s}\sin\frac{n\pi ct}{L}\right)\cos\frac{n\pi x}{L} \quad (29)$$

Similarly, applying the initial conditions (7) to (29), $D_1$ and $D_2$ are obtained as:

$$D_1 = 0 \quad (30)$$

$$D_2 = \frac{-n\pi cW}{\lambda_{n2}(\lambda_{n2}-\alpha_s)L} \quad (31)$$

Thus (29) becomes,

$$y(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\left(\frac{W}{\lambda_{n2}-\alpha_s}\left(\sin\frac{n\pi ct}{L} - \frac{n\pi c}{\lambda_{n2}l}\sin\lambda_{n2}t\right)\right)\cos\frac{n\pi x}{L} \quad (32)$$

Equation (32) is the solution to the distributed load problem when the Euler Bernoulli beam has sliding/guided end conditions.

## 4. IMPOSING THE SECOND NATURAL FREQUENCY ON THE DISTRIBUTED LOAD PROBLEM OF THE EULER BERNOULLI BEAM.

The natural frequency otherwise known as eigen frequency is that frequency at which a system vibrates in the absence of a damping or externally applied force. A system exhibits a harmonic vibration at its second natural frequency and since the Euler-Bernoulli in consideration bears an externally applied force, (that is the distributed load) it implies that a frequency other than the natural frequency is quite expected. Since harmonic vibration provides a basis for the charaterisation and modeling of more complicated vibrations through the application of the Fourier analytical method hence, the need to consider the second natural frequency.

To compute the natural frequency of a simply supported beam with uniformly distributed load *w* per unit length (inclusive of the beam self weight), the formula

$$\gamma_n = \frac{K_n}{2\pi}\sqrt{\frac{EIg}{wl^4}} \qquad (33)$$

where
I = area moment of inertia,
$l$ = length of the beam
$g$ = gravitational acceleration
E = modulus of elasticity
$w$ = weight per unit length of the beam
$K_n$ = constant ( n refers to constant mode of vibration).
The constant for mode of vibration is what differentiates the fundamental natural frequency from all other modes of vibration such as the second, third, fourth and fifth natural frequencies while other parameters remain constant.
For Mode 1 which corresponds to the first or fundamental natural frequency $K_n = 9.87$
For Mode 2 which corresponds to the second natural frequency $K_n$ = 39.5
For Mode 3 which corresponds to the third natural frequency $K_n$ = 88.8
For Mode 4 which corresponds to the fourth natural frequency $K_n$ = 158
For Mode 5 which corresponds to the fifth natural frequency $K_n$ = 247
(Young and Budynas 2002)

### 4.1 Beam with simple supports

For the Euler Bernoulli beam having simple support and traversed by distributed loads, substituting the first natural frequency $\lambda_{n1}$ in equation (21) with the second natural frequency $\gamma_{n1}$ from equation (33), gives,

$$y(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\left(\frac{WL^5(-1)^{n+1}(1-\cos\gamma_{n1}t)}{EIn^5\pi^5} + \frac{W\left(\cos\frac{n\pi ct}{L} - \cos\gamma_{n1}t\right)}{\gamma_{n1}-\alpha_s}\right)\sin\frac{n\pi x}{L} \qquad (34)$$

Equation (34) is the solution of the distributed load problem having the second natural frequency when the beam is simply supported.

### 4.1 Beam with sliding/guided end conditions

For the Euler Bernoulli beam traversed by distributed load with sliding/guided end conditions, substituting the first natural frequency $\lambda_{n2}$ in equation (32) with the second natural frequency $\gamma_{n2}$ from equation (33) yields,

$$y(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\left(\frac{W}{\gamma_{n2}-\alpha_s}\left(\sin\frac{n\pi ct}{L} - \frac{n\pi c}{\gamma_{n2}\,l}\sin\gamma_{n2}t\right)\right)\cos\frac{n\pi x}{L} \qquad (35)$$

Equation (35) is the solution to the distributed load problem having sliding/guided end conditions with the second natural frequency.

## 5. DISCUSSION OF RESULT

It is important to establish the condition under which resonance occurs for the problem discussed. When the natural frequency is equal to the external exciting frequency, then resonance will occur in the system. Resonance at the natural frequency is usually referred to as the first resonance frequency or first critical speed. For the same vibration energy input, the vibration displacement amplitude at one of the higher criticals (such as second or third frequency) is usually smaller than at the first resonance. Due to resonance in the system the amplitude of vibration is maximum and failure may be occurring in the system. Most vibrations are undesirable in machines and structures because they produce increased stresses, energy losses, cause added wear, increase bearing loads, induce fatigue, create passenger discomfort in vehicles and absorb energy from the system. The vibrations produced have the same frequency as the applied force. Hence, to reduce the vibration of the system, one must know the value of the frequency.

For the first natural frequency, equation (21) shows that the simply supported Euler Bernoulli beam traversed by moving distributed loads reaches resonance when,

$$\lambda_{n1} = \alpha_s \quad (36)$$

While for the sliding/guided end condition, the Euler Bernoulli beam traversed by moving distributed loads reaches state of resonance when,

$$\lambda_{n2} = \alpha_s \quad (37)$$

For the second natural frequency, equation (34) shows that the simply supported beam reaches resonance when,

$$\gamma_{n1} = \alpha_s \quad (38)$$

And the beam with sliding/guided end conditions from equation (35) reaches resonance when

$$\gamma_{n2} = \alpha_s \quad (39)$$

## 6. GRAPHS OF DEFLECTION AGAINST FREQUENCY FOR THE EULER BERNOULLI BEAM CARRYING DISTRIBUTED LOAD.

In order to illustrate the forgoing analysis, a mild steel beam is considered, with specifications given in Table 1. Graphs of deflection against frequency are plotted for the steel beam with the aid of Matrix Laboratory MATLAB. This is done for the distributed load problem, when the beam has simple supports and sliding/guided end conditions, respectively.

### 6.1 GRAPHS FOR FIRST NATURAL FREQUENCY, $\lambda_n$

The values of the constants and natural frequency $\lambda_n$ are obtained using Table 1.

**Table 1: Mild Steel Beam Specification**

| | |
|---|---|
| $\rho$ | $7850 kg/m^3$ |
| $E$ | $2.1 \times 10^{11}$ |
| $Length$ | $10m$ |
| $Width$ | $3m$ |
| $Depth$ | $0.4m$ |

where,

$A = 0.4m \times 3m = 1.2m^2$ (ie, cross sectional area of beam)

Length = 10m

I (moment of inertia) = $\frac{\text{width}(\text{depth}^3)}{12} = 0.0192m^4$

$\pi = \frac{22}{7}$,

c = 5m/s

Assumed weight = 20N

$\mu$ (mass per unit length of steel)= 6280kg/m

Figure 6.1 shows the deflection profile of a simply supported Euler Bernoulli beam under the action of distributed load travelling at a constant speed, with first frequency. From the figure, sharp crested troughs are noticed which is reflective of an undamped vibration pattern associated with the first frequency, having the first mode of vibration as the prominent of all.

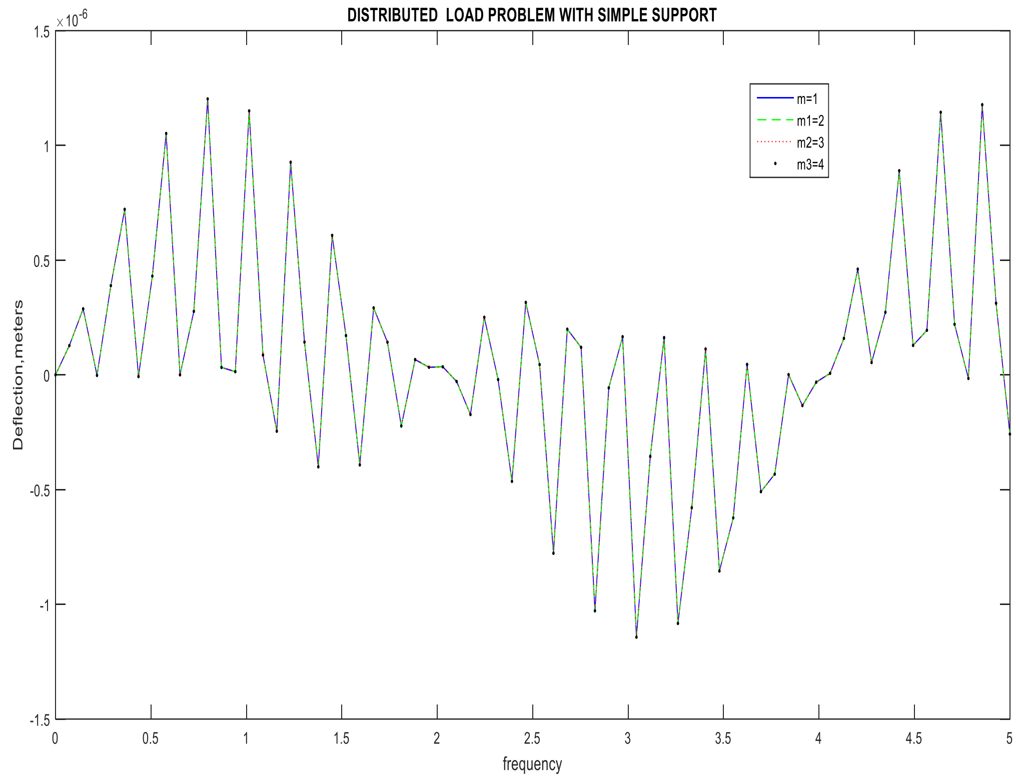


**Figure 6.1** Deflection profile $y(x,t)$ of a simply supported Euler Bernoulli beam under the action of distributed load with first frequency $\lambda_{n1}$.

Figure 6.2 shows the deflection profile for an Euler Bernoulli beam having sliding end conditions under the action of distributed load, with first frequency. Here, higher amplitudes (which in this case are the deflections) can be observed with a more regular and gentle sinusoidal pattern associated with the sliding/guided end condition indicating a more damped vibration mode at first frequency compared to the simple supported Euler beam.

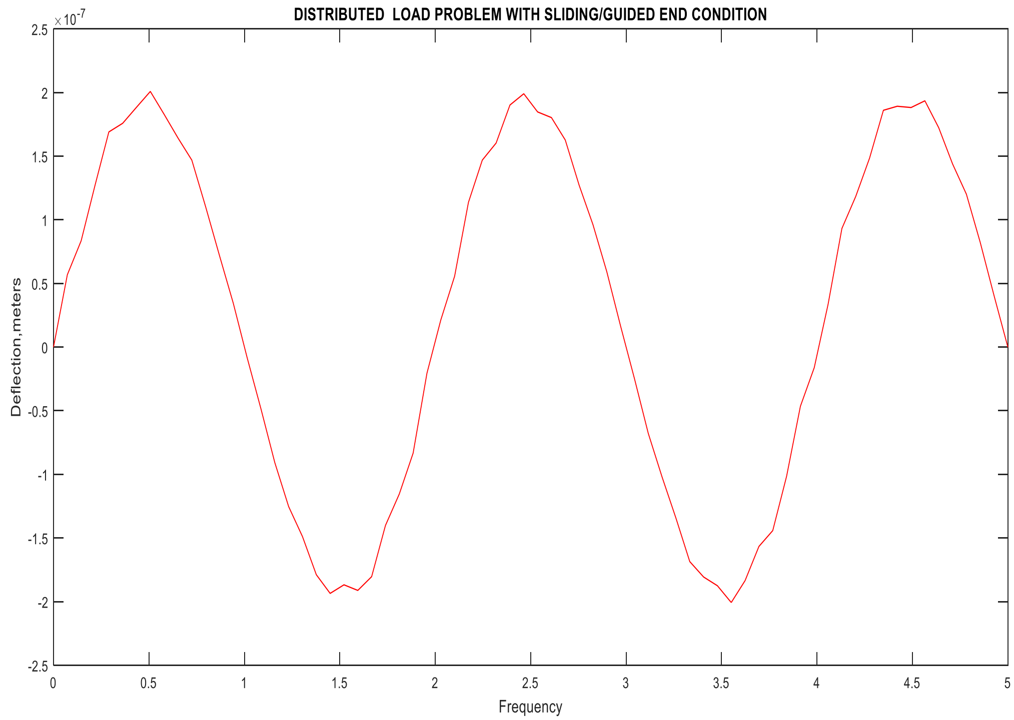


**Figure 6.2:** Deflection profile $y(x,t)$ for an Euler Bernoulli beam having a sliding end conditions under the action of a distributed load with first frequency $\lambda_{n2}$.

## 6.2 GRAPHS FOR IMPOSED NATURAL FREQUENCY, $\gamma_n$

For the imposed natural frequency $\gamma_n$, the following specifications were used:
E = 2.1x10$^{11}$, $\rho$ = 7850kg/m$^3$,
$g$ (gravitational acceleration) = 9.8m/s$^2$,
K (constant mode of vibration) = 9.87,
length = 10m, width= 3m, Depth= 0.4m,
$\mu$(mass per unit length of steel)= 6280kg/m,
Mg(weight)= 20N,
I(moment of inertia)= $\frac{\text{width}\,(\text{depth}^3)}{12} = 0.0192\text{m}^4$,
A= 1.2m$^2$.

Figure 6.3 shows the deflection profile of a Euler Bernoulli beam under action of distributed load with imposed natural frequency having guided/sliding end condition. The figure shows the best vibration pattern: the least amplitude (deflections) induced, most regular and gentle sinusoidal pattern, smoothest and widest troughs. This implies that the imposed natural frequency combined with the guided/sliding end conditions constitute the best for the Euler Bernoulli beam.

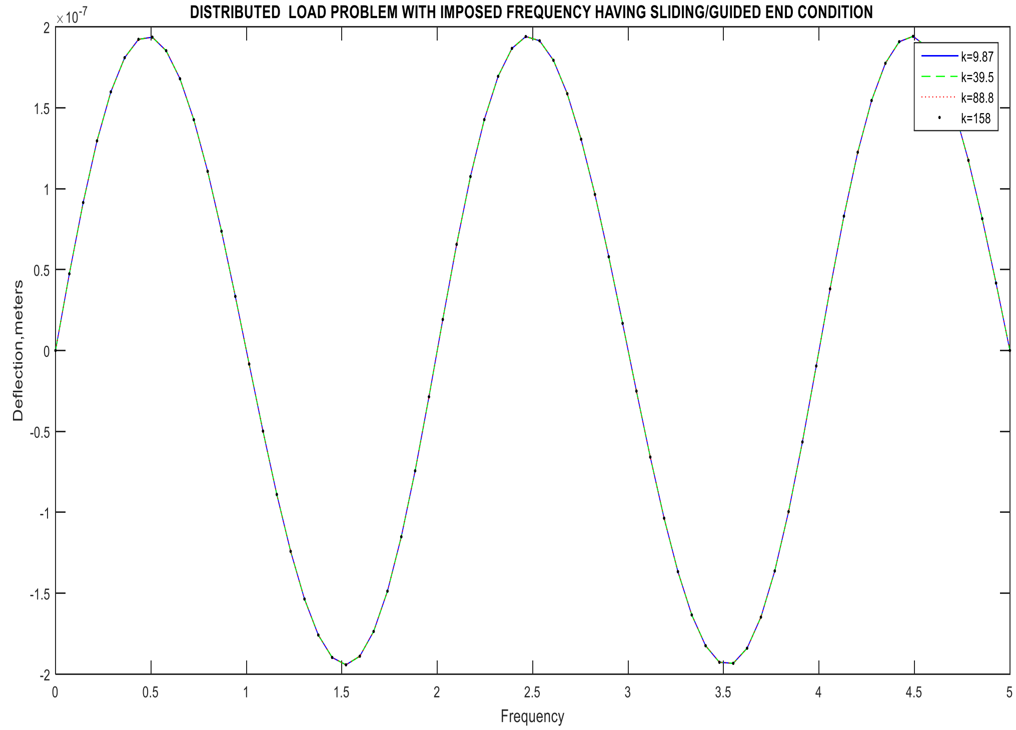


**Figure 6.3:** Deflection profile $y(x,t)$ of an Euler Bernoulli beam with imposed natural frequency $\gamma_{n2}$ having guided/sliding end condition.

While Figure 6.4 depicts the deflection profile of a simply supported Euler Bernoulli beam under the action of distributed loads with imposed natural frequency. Figure 6.4 is in many ways akin to Figure 6.1 only that the frequency of vibration is different yet, this does not provide a safe vibration pattern in that; the amplitudes are irregular and higher (which is precarious for a beam) with very jagged troughs (indicating a very undamped vibration condition).

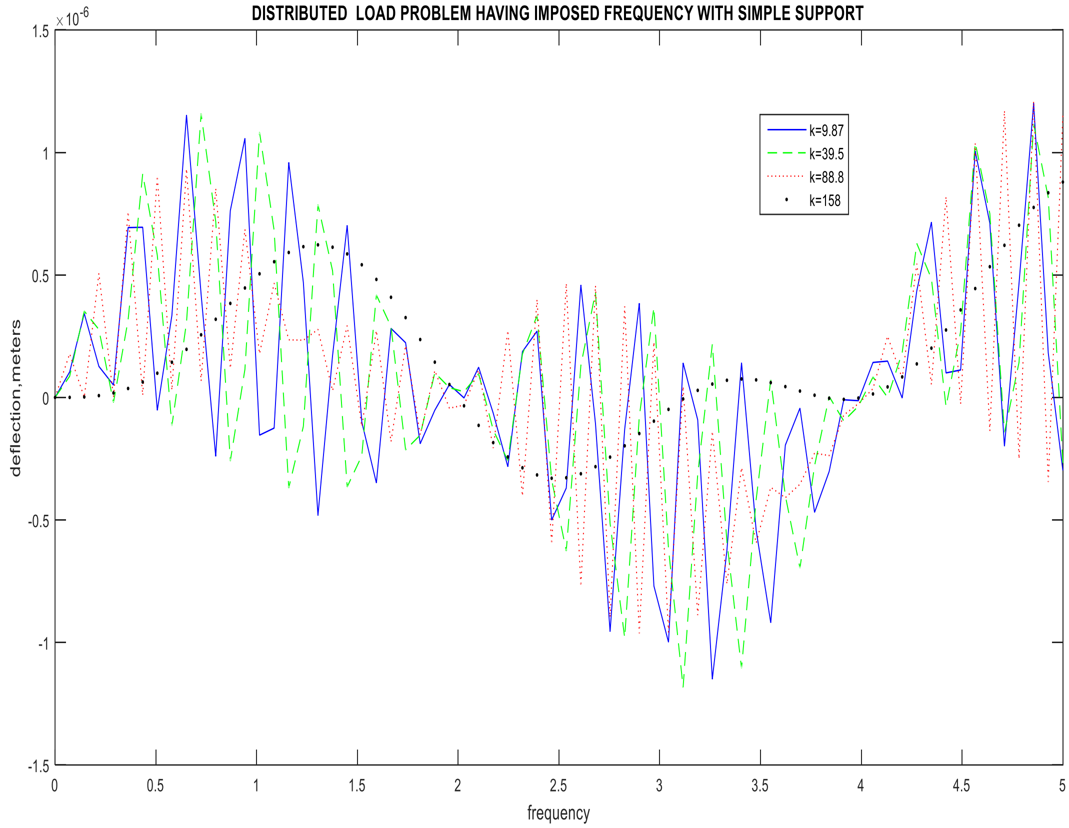


**Figure 6.4:** Deflection profile $y(x,t)$ of a simply supported Euler Bernoulli beam with imposed natural frequency $\gamma_{n1}$ having simple supports.

## CONCLUSION

The study was carried out on the second natural frequency of a Euler Bernoulli steel beam subjected to distributed loads having simply supported and guided/sliding end conditions respectively. The governing equations were tackled using Finite Fourier Sine Transform for the simply supported end condition while Finite Fourier Cosine Transform is employed for the guided/sliding end condition. Graphs were plotted for the first $(\lambda_{n1}, \gamma_{n1})$ and second (imposed) natural frequencies $(\lambda_{n2}, \gamma_{n2})$, respectively. It was observed that the sliding/guided end condition provides the best and safest vibration pattern under (imposed) natural frequencies considering the regular and damped vibration as well as the minimum deflection observed. Conditions under which resonance occurs for the Euler Bernoulli beam were established.

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