ACADEMY JOURNAL OF SCIENCE AND ENGINEERING
Available online @ www.academyjsekad.edu.ng
AJSE 17 (1) 2023
Original Research

INVENTORY MODEL FOR DETERIORATING ITEMS HAVING LINEAR TIME DEPENDENT DEMAND UNDER DOWNSTREAM TRADE CREDIT

Malumfashi, A.A.,[1] Aliyu, Z.H.[2] and Dari, S.[3]

[1]Department of Mathematics and Statistics, Kaduna Polytechnic, Kaduna,
[2]Department of Mathematical Sciences, Nigerian Defence Academy, Kaduna,
[3]Department of Mathematics, Kaduna State University, Kaduna,

,

Abstract

In the literature, either upstream or two-level trade credit was considered in developing inventory models for deteriorating items. Thus, in this study, downstream trade credit is considered only in developing an inventory model for deteriorating items. The demand is assumed to be linearly dependent on time. Cost functions of the model were derived and convexity of the cost functions was established. As an illustration of the performance of the model, numerical examples is given. Sensitivity analysis was carried out to test the importance of the model parameters. From the result of the numerical example, it shows that the total annual relevant costs in the second case is the least and present the optimal case.

Keywords: Deterioration; Inventory; Trade credit; Linear demand; Downstream.

1.0 INTRODUCTION

In the classical EOQ model it was assumed that the retailer must settle the account instantly as soon as the goods ordered are received. But this situation is not always true in reality. In practice, to encourage the retailer to buy more, the supplier will allow a fixed period of time known as trade credit period or permissible delay in payment period for the retailer within which no payment is expected. During the trade credit period, the retailer will makes sales and also earn interest on the sales revenue generated. At the expiration of the period, the retailer will be charged interest on the unsold items. This trade credit, helps the supplier to reduce the on-hand inventory level and as well serves as alternative to price and/or quantity discount.

Goyal (1985) was the first author to established an inventory model with constant demand rate under condition of permissible delay in payment. Pal and Maity (2012) developed an inventory model for deteriorating items with constant

Corresponding author's e-mail: zaharuna@nda.edu.ng *Website: www.academyjsekad.edu.ng*

deterioration rate considering trade credit policy. Shortages are allowed and assumed to be completely backlogged. Sarkar*et al.* (2013) developed an inventory model with finite replenishment and time varying demand rate under trade credit policy. Wou and Zhaob (2015) develops an Economic Order Quantity (EOQ) model for deteriorating items with an inventory-dependent and linearly increasing time-varying demand under trade credit.

In all of the above models that considered trade credit, they assumed that the supplier would offer theretailer a delay period but the retailer wouldnot offer a trade credit to the customers which is defined as one level trade credit or upstream trade credit. In most business transactions, this assumption is debatable. Many researchers have modified this assumption to assume that the retailer can also adopt the trade credit policy to stimulate customer's demand. This situation is defined as two level trade credit. Shou Ting (2015) developed an Economic Production Quantity (EPQ)model for deteriorating items under two level trade credits with upstream full trade credit and also the retailer's charge interest based on non- deteriorated items. Shah and Vaghela (2018) developed an EPQ model for deteriorating items with price dependent demand under two-level trade credit financing. They considered downstream partial trade credit and upstream full trade credit. Yang *et al*. (2020) developed an inventory model to consider the optimal credit period under two level trade credit. They assumed that the demand depends on the price and the credit period provided. All the models that consider trade credit assumed that either the supplier gives trade credit to the retailer (upstream) only or that the supplier gives retailer the permissible delay in payment grace and the retailer pass the same grace to the customer (two - level trade credit).

However, in reality, there exist some situations where the supplier offers no trade credit to the retailer but the retailer offerstrade credit to the customers.Therefore, in this research work downstream trade credit would be considered in developing an inventory system model for deteriorating items where the demand is assumed to be linearly dependent on time.

1.1. Notations

C_1, C_2, C_3- ordering cost,purchasing cost and selling price respectively.

h, θ- holding cost and deterioration cost of an item respectively with $0 < \theta < 1$.

$D\,(t), P$- Demandand production rates respectively, where $P = KD(t), K > 1$

$I(t)$- Inventory level at time t ≥ 0.

t_1, T-Time at which the production stops and length of inventory cycle respectively.

Q- Maximum inventory level when there is no production.

I_c, I_e - Interest charged and interest earn return rates.

N -Trade credit period offered by the retailer to the end customers.

1.2. Assumptions

i. Deterioration rate is constant and instantaneous.
ii. Demand rate is assumed to be linearly depended on time that is $D(t) = a + bt$where a > 0 is the initial demand and b >0is the demand rate.

iii. The production rate is assumed to be greater than the demand rate i.e.,$P > D(t)$.

iv. The supplier offers no trade credit period to the retailer for the goods ordered.

v. The retailer offers trade credit to the customers for a period N.

vi. The customer incurred charges $I_C$as penalty beyond the permissible period N given by the retailer.

vii. $e^{\theta(T-t_1)} \approx 1 + \theta(T - t_1)$

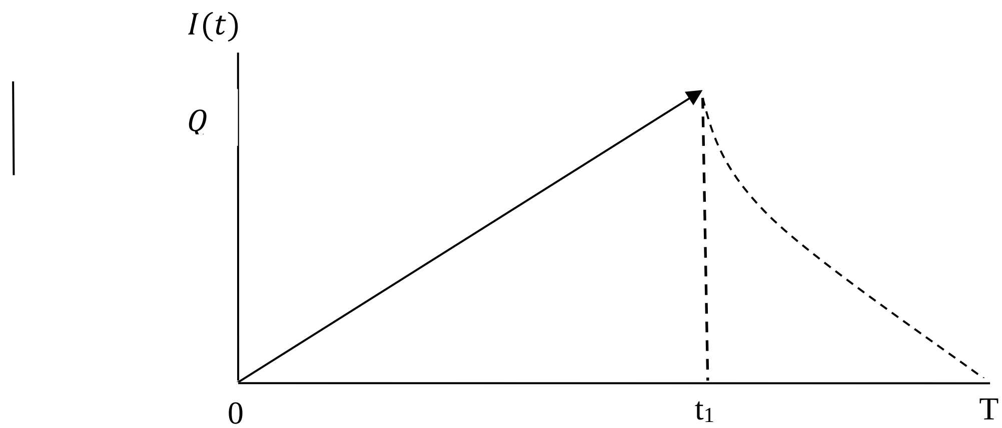

Figure 1: Graphical Representation of the model.

1.3 Model Formulation

During the period$(0, t_1)$,while production is ongoing, inventorylevel reduces due to demand D(t) and deteriorationand it does not affect the inventory buildup because $P > D$. Also, during the period $[t_1, T]$the inventory is depleted due to demand and deterioration.The differential equations governing the instantaneous state of the inventory level are given by

$$\frac{dI_1(t)}{dt} = P - D(t) - \theta I_1(t) \qquad 0 \le t \le t_1 \qquad (1)$$

with initial condition $I_1(0) = 0$

$$\frac{dI_2(t)}{dt} = -D(t) - \theta I_2(t) \qquad t_1 < t \le T \qquad (2)$$

with the condition $I_2(T) = 0$

Substituting$D(t) = a + bt$ and $P = kD(t) = K\,(a + bt)\ in$ equation (1) and (2) we get

$$\frac{dI_1(t)}{dt} + \theta I_1(t) = (K - I)(a + bt) \quad \text{where } K > 1 \qquad (3)$$

$$\frac{dI_2(t)}{dt} + \theta I_2(t) = -(a + bt) \tag{4}$$

Solving equations (3) and (4), we obtain

$$I_1(t) = \frac{(k-1)}{\theta^2}\left[\left(1 - e^{-\theta t}\right)(a\theta - b) + \theta bt\right] \qquad 0 \le t \le t_1 \tag{5}$$

$$I_2 = \frac{1}{\theta^2}\left[(a\theta - b + \theta bT)e^{\theta(T-t)} - (a\theta - b + \theta bt)\right] \quad t_1 < t \le T \tag{6}$$

Equation (5) and (6) gives the instantaneous level of inventory at any time t, $I(t)$ in the interval $[0, T]$:

$$I(t) = I_1(t) + I_2(t) = \frac{(k-1)}{\theta^2}\left[\left(1 - e^{-\theta t}\right)(a\theta - b) + \theta bt\right]$$

$$+ \frac{1}{\theta^2}\left[(a\theta - b + \theta bT)e^{\theta(T-t)} - (a\theta - b + \theta bt)\right] \tag{7}$$

To get the total relevant cost we find the following inventory quantities:

a. Annual Ordering Cost, *OC*

The ordering cost per order is given as c_1 .Hence, the annual ordering cost is

$$OC = \frac{c_1}{T} \tag{8}$$

b. Annual Holding Cost, *H*

The total annual holding cost of inventory in stock during the interval $[0, T]$, says H, is $\frac{h}{T}\int_0^T I(t)dt = \frac{h}{T}\left(\int_0^{t_1} I_1(t)dt + \int_{t_1}^T I_2(t)dt\right)$. Using equations (5) and (6) and after simplifying, it is given as

$$H = \frac{h}{T\theta^2}\left((k-1)\left((a\theta - b)\left(t_1 + \frac{e^{-\theta t_1} - 1}{\theta}\right) + \theta b\frac{{t_1}^2}{2}\right) + \left(-\frac{(a\theta - b + \theta bT)}{\theta}\left(1 - e^{\theta(T-t_1)}\right)\right) - \left(T\left[a\theta - b + \theta b\frac{T}{2}\right] - t_1\left[a\theta - b + \theta b\frac{t_1}{2}\right]\right)\right) \tag{9}$$

c. Annual Deterioration Cost, D_T

Total demand during the interval $[0, T]$ is given as R_T

$$R_T = \int_0^T D(t)dt = at_1 + b\frac{t_1^2}{2} + aT + b\frac{T^2}{2} - at_1 - b\frac{t_1^2}{2} = aT + b\frac{T^2}{2}$$

The maximum inventory level is $I_1(t_1) = Q$.

Using equation (5),

$$I_1(t_1) = Q = \frac{(k-1)}{\theta^2}\left[\left(1 - e^{-\theta t_1}\right)(a\theta - b) + \theta bt_1\right]$$

The number of deteriorated items are $Q - R_T$

Hence, the annual total deterioration cost D_T is given as:

$$D_T = \frac{c_2}{T}(Q - R_T) =$$

$$\frac{c_2(k-1)}{T\theta^2}\left[\left(1 - e^{-\theta t_1}\right)(a\theta - b) + \theta b t_1\right] - \frac{c_2}{T}\left(aT + b\frac{T^2}{2}\right) \quad (10)$$

d. Annual Interest payable by the retailer and Annual Interest earned by the retailer

The annual interest earned and interest paid by retailer are derived under two cases, case 1: when $N < T$ and case 2: when $N \geq T$

Case1: $N < T$ (Trade credit period given to customer is less than the replenishment cycle).

In this case,since supplier offersno trade credit to the retailer, then the annual total interest payable by the retailer says P_R is

$$P_{R1} = 0 \quad (11)$$

For the interest earned; since there are still unsold goods with the customer at the expiration of the trade credit period given, then the retailer may earn interest in two ways (scenarios):

Scenario 1:when partial payment is acceptable by the retailer and it is to be made at $t = N$ and the rest amountat a time after $t = N$

Let's assume the customer pays some amount (partial payment) at $t = N$ and will balance the remaining amount $C_3 - C_R$ after N $(t > N)$.The total amount paid by costumer says C_R during the period $[0, N]$.

$$C_R = C_3\int_0^N D(t)dt = C_3\int_0^N (a + bt)dt = C_3\left(aN + b\frac{N^2}{2}\right)$$

Let the rest amount to be paid to the retailer at time $t = B$ be $(C_3Q - C_R)$, where $(B > N)$. Then total annual interest payable by the customer to the retailer at time $t = B$ for scenario 1 say $E_{11}$is given as

$$E_{11} = \frac{1}{T}((C_3Q - C_R) + I_e(C_3Q - C_R)(B - N)) = \frac{1}{T}(C_3Q - C_R)(1 + I_e(B - N))$$

$$= \frac{C_3}{T}\left(\left(\frac{(k-1)}{\theta^2}\left[\left(1 - e^{-\theta t_1}\right)(a\theta - b) + \theta b t_1\right]\right) - \left(aN + b\frac{N^2}{2}\right)\right)(1 + I_e(B - N)) \quad (12)$$

Therefore, the total relevant cost forscenario 1 = Ordering cost+ Holding cost + Deterioration cost+ Interest payable by the retailer- Interest earned by the retailer

$$C_{11}(T) = OC + H + D_T + P_R - E_{11}$$

Using equations (8), (9), (10), (11) and (12) and simplifying, we get

$$C_{11}(T) = \frac{1}{T}\left(c_1 + \frac{h}{\theta^2}\left((k-1)\left((a\theta - b)\left(t_1 + \frac{e^{-\theta t_1} - 1}{\theta}\right) + \frac{\theta b {t_1}^2}{2}\right) + \left(-\frac{(a\theta - b + \theta bT)}{\theta}\left(1 - e^{\theta(T - t_1)}\right)\right) - \left(T\left[a\theta - b + \theta b\frac{T}{2}\right] - t_1\left[a\theta - b + \theta b\frac{t_1}{2}\right]\right)\right) + \frac{c_2}{\theta^2}\left[(k-1)\left(\left(1 - \right.\right.\right.$$

$e^{-\theta t_1})(a\theta - b) + \theta b t_1\Big) - \Big(aT + b\frac{T^2}{2}\Big)\Big] - C_3\Bigg(\Big(\frac{(k-1)}{\theta^2}\Big[\big(1 - e^{-\theta t_1}\big)(a\theta - b) +$

$\theta b t_1\Big]\Big) - \Big(aN + b\frac{N^2}{2}\Big)\Big)(1 + I_e(B - N))\Bigg)$ (13)

Scenario 2**:** (when partial payment is not acceptable at $t > N$)
In this scenario, full payment is to be made after $t = N$ due to no willingness of the retailer to accept partial payment. Let full payment be made at time $t = B$,where $B > N$.
Thus in this scenario the interest payable by customer says E_{12} to the retailer is given as

$$E_{12} = C_3Q + (C_3QI_e(B - N)) = C_3Q(1 + I_e(B - N))$$

$$= C_3\left(\frac{(k-1)}{\theta^2}\left[\left(1 - e^{-\theta t_1}\right)(a\theta - b) + \theta b t_1\right]\right)(1 + I_e(B - N)) \quad (14)$$

Therefore, the total relevant cost forscenario2 = Ordering cost+ Holding cost + Deterioration cost+ Interest payable by the retailer- Interest earned by the retailer.

$$C_{12}(\mathrm{T}) = OC + H + D_T + P_{R1} - E_{12}$$

Using equations (8), (9), (10), (11) and (14) and simplifying, we have

$C_{12}(T) = \frac{1}{T}\Bigg(c_1 + \frac{h}{\theta^2}\Bigg((k-1)\Bigg((a\theta - b)\Big(t_1 + \frac{e^{-\theta t_1}-1}{\theta}\Big) + \frac{\theta b t_1{}^2}{2}\Bigg) + \Big(-\frac{(a\theta-b+\theta bT)}{\theta}(1 -$

$e^{\theta(T-t_1)})\Big) - \Big(T\Big[a\theta - b + \theta b\frac{T}{2}\Big] - t_1\Big[a\theta - b + \theta b\frac{t_1}{2}\Big]\Big)\Bigg) + \frac{c_2}{\theta^2}\Big[(k-1)\Big(\big(1 -$

$e^{-\theta t_1})(a\theta - b) + \theta b t_1\Big) - \Big(aT + b\frac{T^2}{2}\Big)\Big] - C_3\Bigg(\Big(\frac{(k-1)}{\theta^2}\Big[\big(1 - e^{-\theta t_1}\big)(a\theta - b) +$

$\theta b t_1\Big]\Big)\Big)(1 + I_e(B - N))\Bigg)$ (15)

Case2**:** when $N \geq T$ (when trade credit period offered to the customers exceed the replenishment cycle).
Since the supplier offers no trade credit to the retailer, then the annual total interest payable by the retailer for this case says P_{R2} is

$P_{R2} = 0$ (16)

On the other hand,since $N \geq T$, the customer pays no interest to the retailer. Therefore, the interest earned by the retailer says E_2 is given by

$E_2 = 0.$ (17)

Thus the total relevant cost for case2 = Ordering cost+ Holding cost + Deterioration cost+ Interest payable by the retailer - Interest earned by the retailer.

$$C_2(T) = OC + H + D_T + P_{R2} - E_2$$

Using equations(8), (9), (10), (16) and (17) and simplifying, we see that

$$C_2(T) = \frac{1}{T}\left(c_1 + \frac{h}{\theta^2}\left((k-1)\left((a\theta - b)\left(t_1 + \frac{e^{-\theta t_1}-1}{\theta}\right) + \frac{\theta b t_1{}^2}{2}\right) + \left(-\frac{(a\theta-b+\theta bT)}{\theta}(1 - e^{\theta(T-t_1)})\right) - \left(T\left[a\theta - b + \theta b\frac{T}{2}\right] - t_1\left[a\theta - b + \theta b\frac{t_1}{2}\right]\right)\right) + \frac{c_2}{\theta^2}\left[(k-1)\left((1 - e^{-\theta t_1})(a\theta - b) + \theta b t_1\right) - \left(aT + b\frac{T^2}{2}\right)\right]\right) \quad (18)$$

2 OPTIMALITY CONDITIONS AND ANALYSIS

The objective of the study is to find the optimum cost of maintaining the inventory, by minimizing the total cost functions$C_{11}(T)$, $C_{12}(T)$ and $C_2(T)$,therefore, the necessary conditions to obtain minimum of$C_{11}(\mathrm{T})$, $C_{12}(\mathrm{T})$ and$C_2(\mathrm{T})$ arerespectively given as $\frac{dC_{11}}{dT} = 0$, $\frac{dC_{12}}{dT} = 0$and$\frac{dC_2}{dT} = 0$.

Forcase1 scenario 1,

Differentiating equation (13) with respect to T and setting the result to zero, we get

$$-c_1 + \frac{h}{\theta^2}\left(-(k-1)\left((a\theta - b)\left(t_1 + \frac{e^{-\theta t_1}-1}{\theta}\right) + \frac{\theta b t_1{}^2}{2}\right) + \left(\frac{a\theta-b}{\theta} + \left(-\frac{(a\theta-b+\theta bT)}{\theta} + a\theta T + \theta b T^2\right)e^{\theta(T-t_1)}\right) - \frac{T^2\theta b}{2} - t_1\left[a\theta - b + \theta b\frac{t_1}{2}\right]\right) + \frac{c_2}{\theta^2}\left[(k-1)(-(1 - e^{-\theta t_1})(a\theta - b) - \theta b t_1) - \frac{bT^2}{2}\right] + C_3\left(\left(\frac{(k-1)}{\theta^2}\left[(1 - e^{-\theta t_1})(a\theta - b) + \theta b t_1\right]\right) - \left(aN + b\frac{N^2}{2}\right)\right)(1 + I_e(B - N)) = 0 \quad (19)$$

From assumptions, $e^{\theta(T-t_1)} \approx 1 + \theta(T - t_1)$ and then rearrange (19) in terms of T, we get

$$\left[-c_1 + \frac{h}{\theta^2}\left\{-(k-1)\left((a\theta - b)\left(t_1 + \frac{e^{-\theta t_1} - 1}{\theta}\right) + \frac{\theta b t_1{}^2}{2}\right) - \theta b\frac{t_1{}^2}{2}\right\} + \frac{c_2}{\theta^2}\{(k-1)(-(1 - e^{-\theta t_1})(a\theta - b) - \theta b t_1)\} + C_3\left(\left(\frac{(k-1)}{\theta^2}\left[(1 - e^{-\theta t_1})(a\theta - b) + \theta b t_1\right]\right) - \left(aN + b\frac{N^2}{2}\right)\right)(1 + I_e(B - N))\right] + \left[\frac{h}{\theta}\{bt_1 - a\theta t_1\}\right]T + \left(\frac{h}{\theta^2}\left\{a\theta^2 - b\theta^2 t_1 - \frac{\theta b}{2}\right\} + \frac{c_2}{\theta^2}\left[-\frac{b}{2}\right]\right)T^2 + hbT^3 = 0$$

which can be written as

$$w + xT + yT^2 + zT^3 = 0 \quad (20)$$

where

$$w = \left[-c_1 + \frac{h}{\theta^2}\left\{-(k-1)\left((a\theta - b)\left(t_1 + \frac{e^{-\theta t_1}-1}{\theta}\right) + \frac{\theta b {t_1}^2}{2}\right) - \theta b \frac{{t_1}^2}{2}\right\} + \frac{c_2}{\theta^2}\{(k - 1)\left(-(1-e^{-\theta t_1})(a\theta - b) - \theta b t_1\right)\} + C_3\left(\left(\frac{(k-1)}{\theta^2}\left[(1-e^{-\theta t_1})(a\theta - b) + \theta b t_1\right]\right) - \left(aN + b\frac{N^2}{2}\right)\right)(1 + I_e(B - N))\right],$$

$$x = \left[\frac{h}{\theta}\{bt_1 - a\theta t_1\}\right],$$

$y=\left[\frac{h}{\theta}\{bt_1 - a\theta t_1\}\right]$ and

$$z = hb$$

Lemma 1: if $b = a\theta$, then $(i) x = 0$, $(ii)\ y = 0$, and $(iii)\ w < 0$

Proof: (i)given that

$$x = \left[\frac{h}{\theta}\{bt_1 - a\theta t_1\}\right] = \frac{h}{\theta}\{a\theta t_1 - a\theta t_1\} = 0$$

To see (ii), we have

$$y = \left[\frac{h}{\theta}\{bt_1 - a\theta t_1\}\right] = \frac{h}{\theta}\{a\theta t_1 - a\theta t_1\} = 0$$

For (iii)

$$w = -\left[c_1 + h\left\{(k-1)\left(\frac{a{t_1}^2}{2}\right) + a\frac{{t_1}^2}{2}\right\} + c_2\{(k-1)(at_1)\} + C_3\left(\left(aN + a\theta\frac{N^2}{2}\right) - ((k-1)[at_1])\right)(1 + I_e(B - N))\right] < 0$$

Theorem 1: With the conditions in **lemma 1**, the best cycle length is

$$T^* = \sqrt[3]{-\frac{w}{z}}$$

Proof: from lemma 1, $x = y = 0$, and $w < 0$ and substituting these into (20), we have

$$w + zT^3 = 0$$

$$T^3 = -\frac{w}{z}$$

$$T^* = \sqrt[3]{-\frac{w}{z}}$$

Theorem 2: with the conditions in lemma 1, the total cost function C_{11} is a convex function.

Proof:

To check for the sufficient condition of optimality, taking second derivatives of the function and applying lemma 1, we see that

$$\frac{d^2C_{11}(T)}{dT^2} = \frac{2}{T^3}\left(c_1 + \frac{h}{\theta^2}\left((k-1)\left(\frac{a\theta^2{t_1}^2}{2}\right) + \left(\frac{a\theta^3T^3}{2}\right)e^{\theta(T-t_1)} + \frac{a\theta^2{t_1}^2}{2}\right)\right.$$
$$+ c_2(k-1)(at_1)$$
$$\left. + C_3\left(\left(aN + a\theta\frac{N^2}{2}\right) - ((k-1)[at_1])\right)(1 + I_e(B - N))\right) > 0$$

Therefore, the cost function C_{11} is convex.

To minimize the cost function C_{12}, we use $\frac{dC_{12}}{dT} = 0$, which gives us the value of T for scenario 2 under case 1.
Differentiating equation (15) with respect to T, simplifying and setting the result to zero, we get

$$\left[-c_1 + \frac{h}{\theta^2}\left\{-(k-1)\left((a\theta - b)\left(t_1 + \frac{e^{-\theta t_1}-1}{\theta}\right) + \frac{\theta b{t_1}^2}{2}\right) - \theta b\frac{{t_1}^2}{2}\right\} + \frac{c_2}{\theta^2}\{(k-1)(-(1-e^{-\theta t_1})(a\theta - b) - \theta bt_1)\} + C_3\left(\left(\frac{(k-1)}{\theta^2}[(1-e^{-\theta t_1})(a\theta - b) + \theta bt_1]\right)\right)(1 + I_e(B - N))\right] + \left[\frac{h}{\theta}\{bt_1 - a\theta t_1\}\right]T + \left(\frac{h}{\theta^2}\left\{a\theta^2 - b\theta^2 t_1 - \frac{\theta b}{2}\right\} + \frac{c_2}{\theta^2}\left[-\frac{b}{2}\right]\right)T^2 + hbT^3 = 0$$

(21)

which can be written as

$$w_2 + xT + yT^2 + zT^3 = 0 \qquad (22)$$

where

$$w = \left[-c_1 + \frac{h}{\theta^2}\left\{-(k-1)\left((a\theta - b)\left(t_1 + \frac{e^{-\theta t_1}-1}{\theta}\right) + \frac{\theta b{t_1}^2}{2}\right) - \theta b\frac{{t_1}^2}{2}\right\} + \frac{c_2}{\theta^2}\{(k-1)(-(1-e^{-\theta t_1})(a\theta - b) - \theta bt_1)\} + C_3\left(\left(\frac{(k-1)}{\theta^2}[(1-e^{-\theta t_1})(a\theta - b) + \theta bt_1]\right)\right)(1 + I_e(B - N))\right],$$

$$x = \left[\frac{h}{\theta}\{bt_1 - a\theta t_1\}\right],$$

$y = \left[\frac{h}{\theta}\{bt_1 - a\theta t_1\}\right]$ and

$$z = hb$$

Note that: x and y are both zero from lemma 1
Lemma 2: if $b = a\theta$, then (i) $w_2 < 0$
Proof: (i) given that

$$w_2 = -\left[c_1 + h\left\{(k-1)\left(\frac{at_1{}^2}{2}\right) + a\frac{t_1{}^2}{2}\right\} + c_2\{(k-1)(at_1)\} + C_3\left(\left((k-1)[at_1]\right)\right)(I_e(N-B)-1)\right] < 0$$

Theorem 3: With the conditions in lemma 2, the best cycle length is

$$T^{**} = \sqrt[3]{-\frac{w_2}{z}}$$

Proof: from lemma 2, $x = y = 0, and\ w_2 < 0$ and substituting these into (22), we get

$$w_2 + zT^3 = 0$$

$$T^3 = -\frac{w_2}{z}$$

$$T^{**} = \sqrt[3]{-\frac{w_2}{z}}$$

Theorem 4: with the conditions in lemma 2, the total cost function $C_{12}$is a convex function

Proof:

To check for the sufficient condition of optimality, taking second derivatives of the function and applying lemma 2, we see that

$$\frac{d^2C_{12}(\mathrm{T})}{d\mathrm{T}^2} = \frac{2}{\mathrm{T}^3}\left(\mathrm{c}_1 + \frac{\mathrm{h}}{\theta^2}\left((\mathrm{k}-1)\left(\frac{\mathrm{a}\theta^2\mathrm{t}_1{}^2}{2}\right) + \left(\frac{\mathrm{a}\theta^3\mathrm{T}^3}{2}\right)\mathrm{e}^{\theta(\mathrm{T}-\mathrm{t}_1)} + \frac{\mathrm{a}\theta^2\mathrm{t}_1{}^2}{2}\right) + \mathrm{c}_2(\mathrm{k}-1)(\mathrm{at}_1) + \mathrm{C}_3\left(\left((\mathrm{k}-1)[\mathrm{at}_1]\right)\right)(\mathrm{I_e}(\mathrm{N}-\mathrm{B})-1)\right) > 0$$

Therefore, the cost function $C_{12}$is convex.

Therefore, the necessary condition for minimizing C_2 (t), is $\frac{dC_2}{dT} = 0$

Differentiating equation (18) w. r. t T, simplifying and setting the result to zero, we obtain

$$\frac{dC_2(T)}{dT} = \left[-c_1 + \frac{h}{\theta^2}\left\{-(k-1)\left((a\theta-b)\left(t_1 + \frac{e^{-\theta t_1}-1}{\theta}\right) + \frac{\theta b t_1{}^2}{2}\right) - \theta b\frac{t_1{}^2}{2}\right\} + \frac{c_2}{\theta^2}\{(k-1)(-(1-e^{-\theta t_1})(a\theta-b) - \theta b t_1)\}\right] + \left[\frac{h}{\theta}\{bt_1 - a\theta t_1\}\right]T + \left(\frac{h}{\theta^2}\left\{a\theta^2 - b\theta^2 t_1 - \frac{\theta b}{2}\right\} + \frac{c_2}{\theta^2}\left[-\frac{b}{2}\right]\right)T^2 + hbT^3 = 0 \qquad (23)$$

which can be written as

$$\mathrm{w}_3 + \mathrm{xT} + \mathrm{yT}^2 + \mathrm{zT}^3 = 0$$

where

$$w_3 = \left[-c_1 + \frac{h}{\theta^2}\left\{-(k-1)\left((a\theta-b)\left(t_1 + \frac{e^{-\theta t_1}-1}{\theta}\right) + \frac{\theta b t_1{}^2}{2}\right) - \theta b\frac{t_1{}^2}{2}\right\} + \frac{c_2}{\theta^2}\{(k-1)(-(1-e^{-\theta t_1})(a\theta-b) - \theta b t_1)\}\right],$$

$$x = \left[\frac{h}{\theta}\{bt_1 - a\theta t_1\}\right],$$

$y = \left[\frac{h}{\theta}\{bt_1 - a\theta t_1\}\right]$and

$$z = hb$$

Similarly, from lemma 2, x and y are both $zero$

Lemma 3: if $\mathrm{b} = \mathrm{a}\theta$, then $\mathrm{w}_3 < 0$

Proof: from the assumptions,$e^{\theta(T-t_1)} \approx 1 + \theta(T - t_1)$, therefore,

$$w_3 = -\left[c_1 + h\left\{(k-1)\left(\frac{a{t_1}^2}{2}\right) + a\frac{{t_1}^2}{2}\right\} + c_2\{(k-1)(at_1)\}\right] < 0$$

Theorem 5: With the condition in lemma 3, the best cycle length is

$$T^* = \sqrt[3]{-\frac{w_3}{z}}$$

Proof: from lemma 1,$x = y = 0, and\ w_2 < 0$ andsubstituting these into (23), we have

$$w_3 + zT^3 = 0$$

$$T^3 = -\frac{w_3}{z}$$

$$T^* = \sqrt[3]{-\frac{w_3}{z}}$$

Theorem 6: With the conditions in lemma 3, the total cost function $C_2$is a convex function

Proof:

To check for the sufficient condition of optimality, taking second derivatives of the function and applying lemma 3, we get.

$$\frac{d^2C_2(T)}{dT^2} = \frac{1}{T^3}\left(2c_1 + \frac{h}{\theta^2}\left(2(k-1)\left((a\theta - b)\left(t_1 + \frac{e^{-\theta t_1}-1}{\theta}\right) + \frac{\theta b{t_1}^2}{2}\right) - \frac{2(a\theta-b)}{\theta} + \left(2a - \frac{2b}{\theta} + 2bT - 2a\theta T - \theta bT^2 + a\theta^2T^2 + b\theta^2T^3\right)e^{\theta(T-t_1)} + 2t_1\left[a\theta - b + \theta b\frac{t_1}{2}\right]\right) - \frac{c_2}{\theta^2}\left[2(k-1)\left(-\left(1-e^{-\theta t_1}\right)(a\theta - b) - \theta bt_1\right)\right]\right) > 0$$

Therefore, the cost function $C_2$is convex.

3 NUMERICAL EXAMPLES

Some of the parameters used in this example are adopted from Shah &Vaghela (2018), therefore, $a = 50, b = 5, c_2 = 10, c_1 = 540,\ c_3 = 7\ , I_p = 0.4, k = 2, h = 270, N = 0.2\ \ B = 2,\ t_1 = 1\ and\ I_e = 0.1$

Table 1: Numerical Example

CASE	Sub-case	T (YEARS)	COST (₦)
1	C_{11}	3.603020	21960.73890
2	C_{12}	3.601432	21912.06186
3	C_2	3.606286	22089.57303

3.1 Sensitivity Analysis

To know how the optimal solution is affected by the values of the parameters, we drive the sensitivity analysis of the parameters on the optimal solution by changing each of the parameters by -10%, -20%,+10%, and +20%, taking one parameter at a time and keeping the remaining eleven parameters unchanged.

Table2: Sensitivity analysis result of cases (percentage change in system parameters against values for case 1.1case 1.2 and case 2)

Parameter	% change in parameter	Case 1.1		Case 1.2		Case 2	
		T^*	C_{11}	T^{**}	C_{12}	T^{***}	C_2
a	+20	3.86963439	30573.45932	3.867986	30513.67	3.873315	30733.66
	+10	3.741077288	26122.57154	3.739459	26068.46	3.744561	26266.71
	0	3.603019808	21960.7389	3.601432	21912.06	3.606286	22089.57
	-10	3.453484522	18093.79969	3.451928	18050.34	3.456506	18208.08
	-20	3.289739785	14529.08767	3.288213	14490.61	3.292482	14629.57
b	+20	3.344125782	18757.66037	3.342587	18709.49	3.346969	18883.74
	+10	3.466609491	20245.0278	3.465049	20196.69	3.469657	20372.2
	0	3.603019808	21960.7389	3.601432	21912.06	3.606286	22089.57
	-10	3.756849826	23974.54642	3.755229	23925.3	3.760354	24105.81
	-20	3.932984428	26389.66274	3.931322	26339.53	3.936753	26524.4
C_1	+20	3.605072804	22023.71734	3.603487	21975.07	3.608335	22152.47
	+10	3.604046598	21992.23332	3.60246	21943.57	3.607311	22121.03
	0	3.603019808	21960.7389	3.601432	21912.06	3.606286	22089.57
	-10	3.601992433	21929.23406	3.600404	21880.54	3.60526	22058.11
	-20	3.60096447	21897.71878	3.599375	21849.01	3.604234	22026.64
C_3	+20	3.604670278	21273.72098	3.603084	21225.12	3.607933	21402.39
	+10	3.603845232	21617.2462	3.602258	21568.61	3.60711	21746
	0	3.603019808	21960.7389	3.601432	21912.06	3.606286	22089.57
	-10	3.602194006	22304.19908	3.600606	22255.48	3.605462	22433.12
	-20	3.601367825	22647.62669	3.599779	22598.87	3.604637	22776.63
C_3	+20	3.602365895	21934.9532	3.60046	21876.52	3.606286	22089.57
	+10	3.602692881	21947.84684	3.600946	21894.29	3.606286	22089.57
	0	3.603019808	21960.7389	3.601432	21912.06	3.606286	22089.57
	-10	3.603346676	21973.62939	3.601918	21929.83	3.606286	22089.57
	-20	3.603673485	21986.51831	3.602404	21947.59	3.606286	22089.57
Θ	+20	3.460869787	22594.65584	3.459149	22540.73	3.464499	22733.32
	+10	3.528415883	22332.34131	3.526761	22281.05	3.531878	22466.28
	0	3.603019808	21960.7389	3.601432	21912.06	3.606286	22089.57

	-10	3.685518375	21406.71489	3.684001	21360.64	3.688553	21530.02
	-20	3.776610684	20543.71053	3.775166	20500.22	3.779371	20660.98
K	+20	4.016560371	29863.38216	4.015283	29818.27	4.020749	30047.39
	+10	3.820979322	25963.20799	3.819568	25916.54	3.824746	26120.22
	0	3.603019808	21960.7389	3.601432	21912.06	3.606286	22089.57
	-10	3.354917152	17816.78331	3.353086	17765.42	3.357565	17915.71
	-20	3.063421068	13468.58027	3.061225	13413.38	3.065255	13534.91
h	+20	3.600475014	27002.50901	3.59915	26953.75	3.603201	27131.54
	+10	3.601632185	24481.64985	3.600188	24432.93	3.604604	24610.59
	0	3.603019808	21960.7389	3.601432	21912.06	3.606286	22089.57
	-10	3.604714342	19439.75902	3.602952	19391.13	3.608339	19568.46
	-20	3.606830272	16918.68452	3.60485	16870.12	3.610903	17047.23
N	+20	3.603350894	21970.99052	3.601449	21912.66	3.606286	22089.57
	+10	3.603185474	21965.86838	3.601441	21912.36	3.606286	22089.57
	0	3.603019808	21960.7389	3.601432	21912.06	3.606286	22089.57
	-10	3.602853898	21955.60212	3.601424	21911.76	3.606286	22089.57
	-20	3.602687745	21950.45809	3.601416	21911.46	3.606286	22089.57
B	+20	3.602908992	21956.36889	3.601268	21906.04	3.606286	22089.57
	+10	3.602964401	21958.55392	3.60135	21909.05	3.606286	22089.57
	0	3.603019808	21960.7389	3.601432	21912.06	3.606286	22089.57
	-10	3.603075214	21962.92384	3.601515	21915.07	3.606286	22089.57
	-20	3.603130618	21965.10874	3.601597	21918.08	3.606286	22089.57
t_1	+20	3.877482767	22788.27104	3.876112	22744.59	3.880958	22931.02
	+10	3.742856017	22401.92932	3.741385	22355.89	3.746241	22538.24
	0	3.603019808	21960.7389	3.601432	21912.06	3.606286	22089.57
	-10	3.456962135	21452.82486	3.455238	21401.14	3.460073	21572.96
	-20	3.30335824	20862.85331	3.301469	20807.7	3.306266	20972.81
I_e	+20	3.602920074	21956.8059	3.601284	21906.64	3.606286	22089.57
	+10	3.602969942	21958.77242	3.601358	21909.35	3.606286	22089.57
	0	3.603019808	21960.7389	3.601432	21912.06	3.606286	22089.57
	-10	3.603069673	21962.70535	3.601506	21914.77	3.606286	22089.57
	-20	3.603119537	21964.67176	3.601581	21917.48	3.606286	22089.57

Table3: Sensitivity analysis result of case (percentage change in system parameters against percentage for case 1.1, case 1.2 and case 2)

Parameter	% change in parameter	Case 1.1		Case 1.2		Case 2	
		% in $\boldsymbol{T}^{*}$	% in C_{11}	% in $\boldsymbol{T}^{**}$	% in C_{12}	% in $\boldsymbol{T}^{***}$	% in C_{2}
a	+20	7.399753434	39.21872	7.401324	39.25515	7.404546	39.132
	+10	3.831715824	18.95124	3.83256	18.96852	3.834277	18.91002
	0	0	0	0	0	0	0
	-10	-4.150276553	-17.6084	-4.15125	-17.6237	-4.15329	-17.5716
	-20	-8.694929252	-33.8406	-8.69708	-33.8692	-8.70157	-33.7716
b	+20	-7.185473299	-14.5855	-7.18728	-14.6156	-7.19068	-14.5129
	+10	-3.785999629	-7.81263	-3.78691	-7.82844	-3.78865	-7.77459
	0	0	0	0	0	0	0
	-10	4.269474666	9.170035	4.270443	9.187812	4.272218	9.127541
	-20	9.158001842	20.16746	9.159979	20.20561	9.163639	20.07657
C_1	+20	0.056979862	0.286777	0.057065	0.287561	0.05682	0.284732
	+10	0.028498049	0.143412	0.028545	0.143805	0.028416	0.142389
	0	0	0	0	0	0	0
	-10	-0.028514288	-0.14346	-0.02854	-0.14385	-0.02844	-0.14244
	-20	-0.057044858	-0.28697	-0.05711	-0.28775	-0.05689	-0.28492
C_3	+20	0.045807953	-3.12839	0.045878	-3.135	0.045679	-3.11091
	+10	0.022909225	-1.56412	0.022949	-1.56742	0.022842	-1.55538
	0	0	0	0	0	0	0
	-10	-0.022919713	1.563974	-0.02294	1.567276	-0.02286	1.555233
	-20	-0.045849946	3.127799	-0.0459	3.134402	-0.04573	3.11032
C_3	+20	-0.018149036	-0.11742	-0.02699	-0.16219	0	0
	+10	-0.009073691	-0.05871	-0.01349	-0.08109	0	0
	0	0	0	0	0	0	0
	-10	0.009072058	0.058698	0.013502	0.081074	0	0
	-20	0.018142463	0.117389	0.026992	0.162135	0	0
Θ	+20	-3.945302237	2.886592	-3.95073	2.869051	-3.93166	2.914275
	+10	-2.070594362	1.692122	-2.07338	1.683949	-2.06327	1.705344
	0	0	0	0	0	0	0
	-10	2.289706175	-2.52279	2.292678	-2.51653	2.281205	-2.53311
	-20	4.817927315	-6.45255	4.824022	-6.44321	4.799536	-6.46728
K	+20	11.47761004	35.98532	11.49129	36.08152	11.49281	36.02522
	+10	6.04935654	18.22557	6.056921	18.27521	6.05775	18.24684
	0	0	0	0	0	0	0
	-10	-6.885964246	-18.8698	-6.89576	-18.924	-6.89687	-18.8952

	-20	-14.97629125	-38.6697	-14.9998	-38.7854	-15.0024	-38.7272
h	+20	-0.070629463	22.95811	-0.06335	23.00876	-0.08555	22.8251
	+10	-0.038512785	11.47917	-0.03454	11.50448	-0.04665	11.41271
	0	0	0	0	0	0	0
	-10	0.047030947	-11.4795	0.042207	-11.5048	0.056942	-11.4131
	-20	0.105757514	-22.9594	0.094904	-23.0099	0.128034	-22.8268
N	+20	0.009189137	0.046682	0.000467	0.002749	0	0
	+10	0.004597982	0.023358	0.000238	0.001374	0	0
	0	0	0	0	0	0	0
	-10	-0.004604749	-0.02339	-0.00022	-0.00137	0	0
	-20	-0.009216241	-0.04681	-0.00045	-0.00275	0	0
B	+20	-0.003075639	-0.0199	-0.00457	-0.02749	0	0
	+10	-0.001537793	-0.00995	-0.00228	-0.01374	0	0
	0	0	0	0	0	0	0
	-10	0.001537758	0.009949	0.002297	0.013742	0	0
	-20	0.003075462	0.019898	0.004584	0.027484	0	0
t_1	+20	7.617581174	3.768235	7.626972	3.799384	7.616468	3.809236
	+10	3.881083554	2.008996	3.886038	2.02552	3.880862	2.031112
	0	0	0	0	0	0	0
	-10	-4.053757145	-2.31283	-4.05934	-2.33169	-4.0544	-2.33873
	-20	-8.316955886	-4.99931	-8.32898	-5.03999	-8.31935	-5.05562
I_e	+20	-0.002768066	-0.01791	-0.00411	-0.02474	0	0
	+10	-0.001384011	-0.00895	-0.00205	-0.01237	0	0
	0	0	0	0	0	0	0
	-10	0.001383985	0.008954	0.002068	0.012368	0	0
	-20	0.002767925	0.017909	0.004126	0.024736	0	0

4.0 Discussion of Results

It is observed from table 2 that;

1. As 'b' the demand rate increases, T^*, T^{**},T^{***}, C_{11},$C_{12}$and C_2 decreases. In real life this as expected as the demand rate increase, the optimal replenishment length decreases .If the demand rate is high the retailer will order for more good, this will result in increment of the total inventory cost and also the optimal replenishment length decreasing.
2. As" c_1" increases,T^*, T^{**},T^{***}, C_{11},$C_{12}$and C_2 increasing. In real life situation if the ordering cost is high the retailer will order for more goods and that will result in increasing of replenishment length and total relevant cost.

3. As "θ" increasing T^{*}, T^{**}, T^{***} decreases while C_{11},$C_{12}$and C_2 increases .But in real life situation, if the deterioration rate is high the replenishment length will decrease as a result of deterioration of the goods which also lead to increment of the inventory cost.
4. As k increase T^{*}, T^{**} ,T^{***} ,C_{11},$C_{12}$and C_2 increases. For real life situation if the production rate is high then the replenishment length and total cost is increasing.
5. As N is increasing T^{*}, T^{**} ,T^{***} ,C_{11},$C_{12}$and C_2 increases. For real life situation if the trade credit period is increasing then the replenishment length and total inventory cost increases.
6. As I_e increases T^{*}, T^{**} ,T^{***}increase while C_{11},$C_{12}$and C_2 decreases in real life situation as the interest is increasing the total cost function is decreasing.

It is observed from table 3 that:
a. it is observed that the annual total relevant inventory costs has: (1) high sensitivity to change in parameters a, k and h (11) moderate sensitivity to change in parameters: b, and (111) low sensitivity to change in parameters:t_1
b. It is found that the optimal replenishment length has: (1) high sensitivity to change in parameters c_1 *and* h (11) moderate sensitivity to change in parameter a, and (111) low sensitivity to change in parameters c_3, B, N.

5.0 CONCLUSION

The model modified Shah and Vaghela's (2018) model by developing an inventory model for deterioration items incorporating the condition of permissible delay in payment at downstream level. Based on the sensitivity analysis, it is observed from table 2 that when the production rate and ordering cost are high then the replenishment cycle length and cost function increases. Also, if the trade credit period increases, then the total inventory cost and the replenishment cycle length increase. Again, when there is an increase in the demand rate then the total inventory relevant cost and the replenishment cycle length decreases.

6.0 RECOMMENDATION

As the model presented incorporated linearly dependent demand and one level trade credit with constant deterioration,it is recommended that the future study will further incorporate the proposed model into more realistic assumptions, such as quadratic demand, an upstream trade credit policy, two level trade credit and also incorporating shortages, multiple retailers and multiple customers and so on.

REFERENCE

Goyal,S.K. (1985).Economic Order Quantity under Conditions of Permissible Delaying Payments.*Journal of the Operational Research Society*,36{4}: 335-338

Lee, Y.P. & Dye, C.Y. (2012).Aninventory model for deteriorating items under stock dependent demand and controllable deterioration rate.*Computer and IndustrialEngineering*, Vol. 63, No. 2, pp. 474–482.

Pal,M &Maity, H.K (2012).An inventory model for deteriorating items with permissible delay in payment and inflation under price dependent demand. *Pakistan journal of statistics and operational research*, 583-592.

Sarkar *etal.*, (2013). Economic Production Quantity models for deteriorating items with price and stock-dependent demand. *Computers and Operational Research*. vol. 32, pp. 297-308.

Shou Ting P. (2015). The EPQ model with deteriorating items under two levels of trade credit in a supply chain system. *Journal of industrial and managementoptimization* 11{2} 479-492

Shah &Vaghela (2018).Retailer's decision for ordering and credit policies for deterioration items.*Applied mathematical and computational*. 7, page 203-206

Within H.P. (1957). An EOQ model for Weibull deteriorating items with power demand and partial backlogging.*Int. Journal of Contemporary MathematicsSciences*, 5, 2010, no. 38 pg. 1895-1904.

Wu &Zhaob (2015).EOQ for deteriorating items with an inventory-dependent and linearly increasing time varying demand under trade credit.*Scientiairanica*, 22{6},2558-2570

Yang*etal*.(2020).Optimal credit period under two levels of trade credit. *American Institute of Mathematics Science* vol. 164pp 1753-1767.