Dynamic Response of Prestressed Tapered Timoshenko Beams to Uniform Partially Distributed Moving Loads

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Abstract

The dynamics of a prestressed variable cross-section Timoshenko beam subjected to a moving partially distributed load is investigated. The Finite element method with Lagrange interpolation functions and reduced integration element were used to model the structure. The Newmark numerical method of integration was used to solve the resulting semi-discrete time dependent equations to obtain the desired responses. The effects of the prestress, moving load's velocity, moving load's length, and boundary conditions on the dynamic characteristic of the beams were investigated and the results presented graphically.

Key words: Prestressed; Tapered Beam; Equal Interpolation; Weak Form.

Mathematics Subject Classification 2010: 70JXX, 74S05.

1. INTRODUCTION

Prestressing involves the application of an initial compressive load on a structure to reduce or eliminate the internal tensile stresses that may be caused by imposed loads or by load-independent effects such as temperature changes or shrinkage. The prestressing of beams, done to improve the overall performance of the structure, has been widely employed in important civil engineering structures such as bridges. Such a bridge is modeled with an axial load, and its dynamic characteristics have been investigated. Fryba [5] studied the free and forced vibration of a simply-supported beam subject to an axial force and a moving force.

Bokaian [2, 3] presented the influence of a constant axial compressive and tensile force on natural frequencies and normal modes of a uniform single-span beam with different boundary conditions. Law and Lu [12] have numerically shown some effects of axial prestressing upon time responses. Saiidi et al. [15] determined the natural frequencies of a prestressed concrete bridge using a simply supported axially compressed beam. The results indicate that an increase in the magnitude of the prestressed force reduces the natural frequencies of prestressed beams. Dallasta and Dezi [4] indicated that the effect of the prestress force on the beam bending vibration frequencies is negligible based on a linear model. Kanaka and Venkateswara [9] have showed that the prestress force reduces the natural frequency of the lower modes, based on a Rayleigh-Ritz formulation that describes prestressing as an external axial compressive force only. Miyamoto et al. [13] dealt with the dynamic behavior of a prestressed beam strengthened with external tendons, where they considered the change in the tendon force along with the compressive force effect, by an incremental formulation of the equations of motion of the beam. Their result showed that the calculated natural frequencies tend to decrease as the amount of the prestressing force increased. Kerr [10] studied experimentally and analytically the dynamic response of a prestressed beam. It was found that the magnitude of the prestress force for a cable that passes through the centroid of the beam cross-section has no effect upon the dynamic response of the beam.

Hamed and Frostig [6] studied the effect of the magnitude of the prestress force on the natural frequencies of prestressed beams, and revealed that the prestress force does not affect the dynamic behavior of general prestressed beams, and reveals that the natural frequencies of bonded prestressed beams can be determined through a linear elastic beam theory with an equivalent moment of inertia of the composite section, while those of the unbonded prestressed beams can be determined by the proposed model. We note here that no external load of any sort was involved in his study.The researches so far mentioned were all based on the Euler-Bernoulli beam theory. Prestressed studies involving Timoshenko beam theory were much fewer. Auciello and Ercolano [1] proposed a general technique for determining the natural frequencies of non-uniform Timoshenko beams. Kocaturk and Simsek [11] analysed the dynamic responses of a damped Timoshenko beam under the combination of an eccentric compressive load and a moving harmonic force. Jiang and Ye [8] analysed the free vibration and transient wave in a prestressed Rayleigh-Timoshenko beam subjected to arbitrary transverse forces using the method of reverberation-ray matrix. Their results showed that frequencies decreased with the prestress.

The objective of this paper is to investigate the dynamic response of prestressed tapered Timoshenko beams to uniform partially distributed moving loads. The prestress is assumed to result from the initial loading by axial forces. Regarding the above cited references, two unique features are considered in the present work. Firstly the beam is semi-tapered; meaning that one dimension of the cross-section gradually reduces in length according to the taper-ratio by Hsu et al [7]. Secondly, the prestress is assumed to vary spatially.

With a literature survey, and to the best of the authors' knowledge, no work was found reported on the dynamic response of a prestressed tapered Timoshenko beam subject to partially distributed moving load. In the current study, we evaluate the dynamic response of a prestressed non-uniform Timoshenko beam under a uniform partially distributed moving load, using the finite element method with the Lagrange interpolation function. Firstly the non-uniform continuous beam was replaced by a discrete system made up of beam elements. The semi-discrete, time dependent elemental and overall stiffness, mass, and centripetal matrices as well as the elemental and overall load vectors were then derived with the aid of the Rayleigh-Ritz method. Newmark numerical integration method was used to obtain the desired responses from the semidiscrete set of integral equations.

Following this introduction, the remainder of the paper is organized as follows. Section 2 presents the mathematical formulation of the problem, while Section 3 involves the first part of the finite element analysis, the spatial approximation. The assembly of the element equations and the imposition of the boundary conditions are also included in Section 3. In Section 4, the time approximation is carried out as the solution of element equations. Numerical examples are given in Section 5, and in Section 6 the concluding remarks brings the discussion to an end.

2. FINITE ELEMENT FORMULATION

Consider a 2-node tapered beam element *i,j* undergoing a compressive axial force *Q* (prestressed) shown in Figure 1. In the figure, *l*, *A*(*x*), *I*(*x*) are the element length, cross-sectional area, and moment of inertia, respectively.

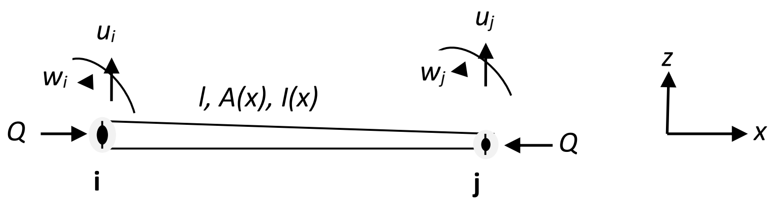

Figure 1. A two-node prestressed beam element

The element has two degrees of freedom at each node, a lateral translation$u$, and a rotation about an axis normal to the plane (*x*, *z*), *w*. Thus the vector of nodal displacements contains four components

$$v = \left\{u_i, w_i, u_j, w_j\right\}^T \tag{1}$$

where (and hereafter) the superscript ^{T} refers to the transpose of a vector or a matrix.

To derive the stiffness, centripetal, and mass matrices for the finite element analysis, an interpolation scheme is required. Using simple linear functions for both the lateral displacement *u* and rotation *w*, that have been widely adopted [16], are possible. However, a special technique should be adopted to prevent a possible *shear locking* problem [17]. Reddy demonstrated that a Timoshenko beam element formulated in the context of the *equal* interpolation, with *reduced* integration of the shear stiffnesses, poses many advantages, including the absence of shear locking. The present study adopted this equal interpolation approach and we obtained the approximation functions as:

$$\left.\begin{aligned} u(x,t) &= \sum_{j=1}^{3}\Phi_j(x)u_j(t) = [\Phi]\{u\} \\ w(x,t) &= \sum_{j=1}^{3}\Psi_j(x)w_j(t) = [\Psi]\{w\} \end{aligned}\right\} \quad (2)$$

where Φ_j and Ψ_j, $j = 1,2,3$ are the Lagrange quadratic approximation functions for $u(x)$ and $w(x)$ respectively.The detail of the expressions for Φ_j and Ψ_j, $j = 1,2,3$ are given by the equations (21) and (22) in the Appendix.

Having derived the interpolation functions, the stiffness, centripetal, and mass matrices can be developed by employing the Ritz technique.

Consider the equation of a pre-stressed tapered Timoshenko beam carrying a load moving at a specified speed

$$\begin{aligned} &\frac{\partial}{\partial x}\left[kGA\left(-\frac{\partial u}{\partial x}+w\right)\right]-\frac{\partial}{\partial x}\left(S\frac{\partial u}{\partial x}\right)-\rho A\frac{\partial^2 u}{\partial t^2}-q(x,t)=0 \\ &-\frac{\partial}{\partial x}\left(EI\frac{\partial w}{\partial x}\right)+kGA\left(-\frac{\partial u}{\partial x}+w\right)+\rho I\frac{\partial^2 w}{\partial t^2}+\rho_q I_q\frac{d^2 w}{dt^2}D=0 \end{aligned} \quad (3)$$

where $u(x,t)$ is the deflection of the beam axis, and $w(x,t)$ is the rotation of its cross-section. ρ and ρ_q are the respective densities of the beam and the load; while I and I_q are the corresponding moments of inertia of their cross-sectional areas, respectively. $A(x)$ is the cross-sectional area of the beam; E - the elastic modulus; G - the shear modulus; k - the shear coefficient; $q(x,t)$ - the distributed load; $S(x)$ is the compressive axial force; t is time, and x is the position coordinate in the axial direction $[x \in (0, L)]$, and L is the length of the beam.

The boundary conditions for a simply supported, clamped, and cantilever beam are respectively

$$
\begin{aligned}
&u(0,t)=u(L,t)=0 \;\; ; \;\; EIw_x(0,t)=EIw_x(L,t)=0, \\
&u(0,t)=u(L,t)=0 \;\; ; \;\; w(0,t)=w(L,t)=0, \\
&u(0,t)=w(0,t)=0 \;\; ; \;\; EIw_x(L,t)=kGA(w_x+w)(L,t)=0,
\end{aligned} \tag{4}
$$

For uniformly distributed load $q(x,t)$, we have

$$
q(x,t)=\frac{1}{\varepsilon}\left[-pg-p\left(\frac{d^2u}{dt^2}\right)\right]D, \tag{5}
$$

where the factor *D* and total derivatives in (3) and (5) are:

$$
\begin{aligned}
&D=H\left[x-\xi+\frac{\varepsilon}{2}\right]-H\left[x-\xi-\frac{\varepsilon}{2}\right], \quad \frac{\varepsilon}{2}\le t\le\frac{L}{v} \\
&\frac{d^2u}{dt^2}=\frac{\partial^2u}{\partial t^2}+2v\frac{\partial^2u}{\partial x\partial t}+v^2\frac{\partial^2u}{\partial x^2} \\
&\frac{d^2w}{dt^2}=\frac{\partial^2w}{\partial t^2}+2v\frac{\partial^2w}{\partial x\partial t}+v^2\frac{\partial^2w}{\partial x^2}
\end{aligned} \tag{6}
$$

p is the mass of the load, and g is the acceleration due to gravity, ε is the load's length, ξ is the distance covered by the load, v is the moving speed of the load, and $H(x)$ is the Heaviside function.
Using (5) and (6) in (3), we obtain

$$
\begin{aligned}
&\frac{\partial}{\partial x}\left[kGA\left(-\frac{\partial u}{\partial x}+w\right)\right]-\frac{\partial}{\partial x}\left(S\frac{\partial u}{\partial x}\right)+\rho A\frac{\partial^2u}{\partial t^2} \\
&\qquad -\frac{1}{\varepsilon}\left[-pg-p\left(\frac{\partial^2u}{\partial t^2}+2v\frac{\partial^2u}{\partial x\partial t}+v^2\frac{\partial^2u}{\partial x^2}\right)\right]\left[H\left(x-\xi+\frac{\varepsilon}{2}\right)-H\left(x-\xi-\frac{\varepsilon}{2}\right)\right]=0 \\
&-\frac{\partial}{\partial x}\left(EI\frac{\partial w}{\partial x}\right)+kGA\left(-\frac{\partial u}{\partial x}+w\right)+\rho I\frac{\partial^2w}{\partial t^2} \\
&\qquad +\rho_qI_q\left(\frac{\partial^2w}{\partial t^2}+2v\frac{\partial^2w}{\partial x\partial t}+v^2\frac{\partial^2w}{\partial x^2}\right)\left[H\left(x-\xi+\frac{\varepsilon}{2}\right)-H\left(x-\xi-\frac{\varepsilon}{2}\right)\right]=0
\end{aligned} \tag{7}
$$

The moment of inertia $I(x)$ and area of beam cross-section of the beam $A(x)$ in (7) are defined respectively by [7] as

$$I(x) = I_0\left[1-\beta_b\frac{x}{L}\right]\left[1-\beta_h\frac{x}{L}\right]^3$$
$$A(x) = A_0\left[1-\right]\left[1-\beta_h\frac{x}{L}\right] \tag{8}$$

where I is the variable moment of inertia of the beam, A is the variable area of beam cross-section, and L is the length of the beam. A_0 and I_0 are the cross-sectional area and moment of inertia at x=0. $\beta_b = 1-\alpha_b$ and $\beta_h = 1-\alpha_h$ are functions of the taper ratios of the beam α_b and α_h respectively.

Following the work of Kien [18] we define the prestress parameter in (7), with axial force S_1, as:

$$S(x) = \frac{L^2}{EI(x)}S_1 \tag{9}$$

The weak forms of the equations (7) over a typical element are

$$\int_0^{l_e}\left[kGA\frac{dR_1}{dx}\left(\frac{\partial u}{\partial x}+w\right)+\frac{dR_1}{dx}S\frac{\partial u}{\partial x}-R_1\rho A\frac{\partial^2 u}{\partial t^2}\right]dx+\frac{pg}{\varepsilon}\int_{\xi-\frac{\varepsilon}{2}}^{\xi+\frac{\varepsilon}{2}}R_1dx+\frac{p}{\varepsilon}\int_{\xi-\frac{\varepsilon}{2}}^{\xi+\frac{\varepsilon}{2}}\frac{\partial^2 u}{\partial t^2}R_1dx$$
$$+\frac{2pv}{\varepsilon}\int_{\xi-\frac{\varepsilon}{2}}^{\xi+\frac{\varepsilon}{2}}\frac{\partial^2 u}{\partial x\partial t}R_1dx+\frac{pv^2}{\varepsilon}\int_{\xi-\frac{\varepsilon}{2}}^{\xi+\frac{\varepsilon}{2}}\frac{\partial^2 u}{\partial x^2}R_1dx-R_1(l_e)Q_3^e+R_1(0)Q_1^e=0 \tag{10}$$

$$\int_0^{l_e}\left[EI\frac{dR_2}{dx}\frac{\partial w}{\partial x}+R_2kGA\left(\frac{\partial u}{\partial x}+w\right)+R_2\rho I\frac{\partial^2 w}{\partial t^2}\right]dx+\rho_qI_q\int_{\xi-\frac{e}{2}}^{\xi+\frac{\varepsilon}{2}}\frac{\partial^2 w}{\partial t^2}R_2dx$$
$$+2v\rho_qI_q\int_{\xi-\frac{e}{2}}^{\xi+\frac{\varepsilon}{2}}\frac{\partial^2 w}{\partial x\partial t}R_2dx+v^2\rho_qI_q\int_{\xi-\frac{e}{2}}^{\xi+\frac{\varepsilon}{2}}\frac{\partial^2 w}{\partial x^2}R_2dx-R_2(l_e)Q_4^e+R_2(0)Q_2^e=0 \tag{11}$$

In (10) and (11), $R_1(x)$ and $R_2(x)$ are the weight functions and $Q_i^e\,(i=1,2,3,4)$ as defined in (12), are the shear forces and bending moments respectively at the boundaries.

$$Q_1^e = \left[kGA\left(\frac{\partial u}{\partial x} + w\right) + S\,\frac{\partial u}{\partial x} \right]_{x=0} \quad ; \quad Q_2^e = \left[EI\,\frac{\partial w}{\partial x} \right]_{x=0}$$
$$Q_3^e = \left[kGA\left(\frac{\partial u}{\partial x} + w\right) + S\,\frac{\partial u}{\partial x} \right]_{x=l_e} \quad ; \quad Q_4^e = \left[EI\,\frac{\partial w}{\partial x} \right]_{x=l_e} \tag{12}$$

Now, employing the Ritz technique, where the weight functions $R_1(x)$ and $R_2(x)$ are defined for each of the approximation function $\Phi(x)$ and $\Psi(x)$ respectively. These, together with (2) in (10) and (11) yields the set of equations of motion for a typical element,

$$\sum_{j=1}^{3}\left(K_{ij}^{*} + K_{ij}^{**} + K_{ij}^{\circ}\right)u_j + \sum_{k=1}^{3} K_{ik} w_k + \sum_{j=1}^{3}\left(M_{ij}^{*} + M_{ij}^{**}\right)\ddot{u}_j + \sum_{j=1}^{3} C_{ij}\dot{u}_j + F_i^1 = 0 \tag{13}$$

$$\sum_{k=1}^{3}\left(K_{ik}^{*} + K_{ik}^{**}\right)w_k + \sum_{j=1}^{3} K_{ij} u_j + \sum_{k=1}^{3}\left(M_{ik}^{*} + M_{ik}^{**}\right)\ddot{w}_k + \sum_{k=1}^{3} C_{ik}\dot{w}_k + F_i^2 = 0 \tag{14}$$

Where in (13) and (14),the terms *K*, *M*, *C*, and *F*, along with their corresponding superscripts and subscripts are defined in (23) to (27) in the Appendix. They are the element stiffness matrices, element centripetal matrices, the element mass matrices, and the element applied forces vector {*f*} plus the element internal generalized forces vector {*Q*}.

3. GOVERNING EQUATION AND ITS SOLUTION

Consider a prestressed tapered Timoshenko beam carrying a load moving at a specified speed, travelling along the beam from left to right. Following the standard procedure of the finite element method, the beam is discretized into a number of finite elements. The above equations of motion of the beam (13) and (14), can be written in terms of the finite element analysis as

$$KV + C\dot{V} + M\ddot{V} = F \tag{15}$$

where *K*, *C*, and *M*, are the structural stiffness, centripetal, and mass matrices, respectively. These matrices are obtained by assembling the corresponding element matrices in the standard way of the finite element method; V , $\dot{V} = \partial V / \partial t$ and $\ddot{V} = \partial^2 V / \partial t^2$ are the vectors of structural nodal displacements, velocities, and accelerations, respectively; *F* is the addition of the assembled element applied forces vector {*f*} and the element internal generalized forces vector {*Q*}.

The system of equation (15) is solved by the Newmark [19, 20] direct integration method using both the linear and average acceleration formulae.

The displacement $V(t)$ the velocity $\dot{V}(t)$ and the acceleration $\ddot{V}(t)$ in (15) are defined by the Newmark equations:

$$V_{s+1} = V_s + \Delta t \dot{V}_s + \left(\tfrac{1}{2} - \beta\right)(\Delta t)^2 \ddot{V}_s + \beta (\Delta t)^2 \ddot{V}_{s+1} \tag{16}$$

$$\dot{V}_{s+1} = \dot{V}_s + \frac{\Delta t}{2}\left(\ddot{V}_s + \ddot{V}_{s+1}\right) \tag{17}$$

The parameter β in (16) defines the variation of acceleration over a time step and determines the stability and accuracy characteristics of the method. The notations used[19] are as defined in (18):

$$V_s = V \text{ at time } t_s \,;\; V_{s+1} = V \text{ at time } t_{s+1} \,;\; \Delta t = t_{s+1} - t_s \tag{18}$$

Writing (17) for $\ddot{V}_{s+1}$ and substituting it into (16), applying it to (15), and collecting like terms results in:

$$\hat{K}_{s+1} V_{s+1} = \hat{F}_{s,s+1} \tag{19}$$

where

$$\left.\begin{aligned}&\hat{K}_{s+1} = \left(K_{s+1} + a_3 M_{s+1} + a_6 C_{s+1}\right)\\&\hat{F}_{s,s+1} = F_{s+1} + M_{s+1}\left(a_3 V_s + a_4 \dot{V}_s + a_5 \ddot{V}_s\right) + C_{s+1}\left(a_6 V_s + a_5 \dot{V}_s + a_7 \ddot{V}_s\right)\\&a_3 = \frac{1}{\beta(\Delta t)^2}\ ,\ a_4 = a_3 \Delta t\ ,\ a_5 = \frac{1}{2\beta} - 1\ ,\ a_6 = \frac{1}{2\beta(\Delta t)}, a_7 = \left(\frac{1}{4\beta} - 1\right)\Delta t\end{aligned}\right\} \quad (20)$$

4. NUMERICAL INVESTIGATIONS

Using the finite element formulated in Section 2 and the numerical algorithm described in Section 3, a computer code in Matlab was developed and used in the dynamic analysis. To investigate the dynamic response, the beam with the following geometry and material data was adopted herewith.

L = 17.5m; ρ=2400kgm^{-3}; E = 2.02×$10^{11}Nm^{-2}$; G=7.7×$10^{10}Nm^{-2}$; k = 5/6; ρ_q = 240kgm^{-3}; I_q=0.0012m^4; p = 1062kg; v = 30ms^{-1}; ε = 0.2m; g = 9.8ms^{-2}. The magnitude of the prestress S = 450000kN will be subjected to variations.
The length of each element is given as:
L_1 = 1.25m, L_2 = 1.25m, L_3 = 1.5m, L_4 = 1.5m, L_5 = 1.75m,
L_6 = 1.75m, L_7 = 2m, L_8 = 2m, L_9 = 2.25m, L_{10} = 2.25m.
The cross-section of the beam is such that its width is uniform from end-to-end, and is given as 0.41m. The non-uniform (tapered) nature of the beam is determined by its varying depth(height), which is given from left to right as:
H_0 = 0.52m, H_1 = 0.5m, H_2 = 0.48m, H_3 = 0.46m, H_4 = 0.44m, H_5 = 0.42m, H_6 = 0.4m,
H_7 = 0.38m, H_8 = 0.36m, H_9 = 0.34m, H_{10} = 0.32m.
Three types of boundary conditions, namely simply supported (SS), clamped (CC), and cantilevered are considered.
The beam's cross-sectional area $A(x)$ and moment of inertia $I(x)$ are calculated using (8). The varying value of the prestress, $S(x)$ is calculated by (9).
The computation is performed with the beam discretized into ten unequal elements.

4.1 Effect of the prestress on the dynamic response

A change was observed in the deflection and rotation of the beam when the prestress was increased from 0KN to 450000KN, 450000KN to 900000KN, and 900000KN to 1350000KN. There was a gradual reduction in the deflection and rotation responses when the prestress was increased. The result is shown in Figure 2.

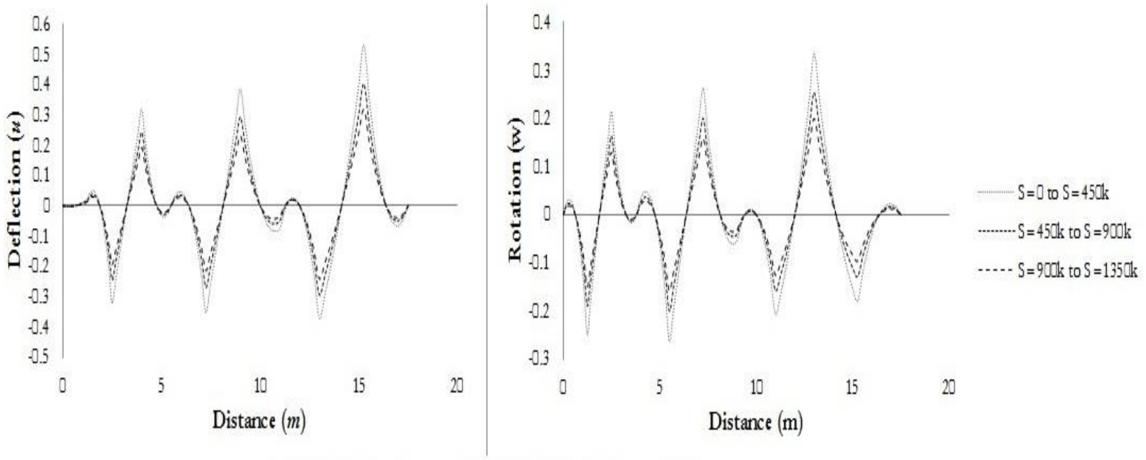

Fig. 2 : Effect of the prestress on the deflection and rotation of the beam.

4.2 Effect of the load's moving velocity on the dynamic response

The effect of load's velocity on the dynamic response of the beam under consideration was increased as the moving velocity of the load increased from 1.3m/s to 1.7m/s. The result is shown in Figure 3.

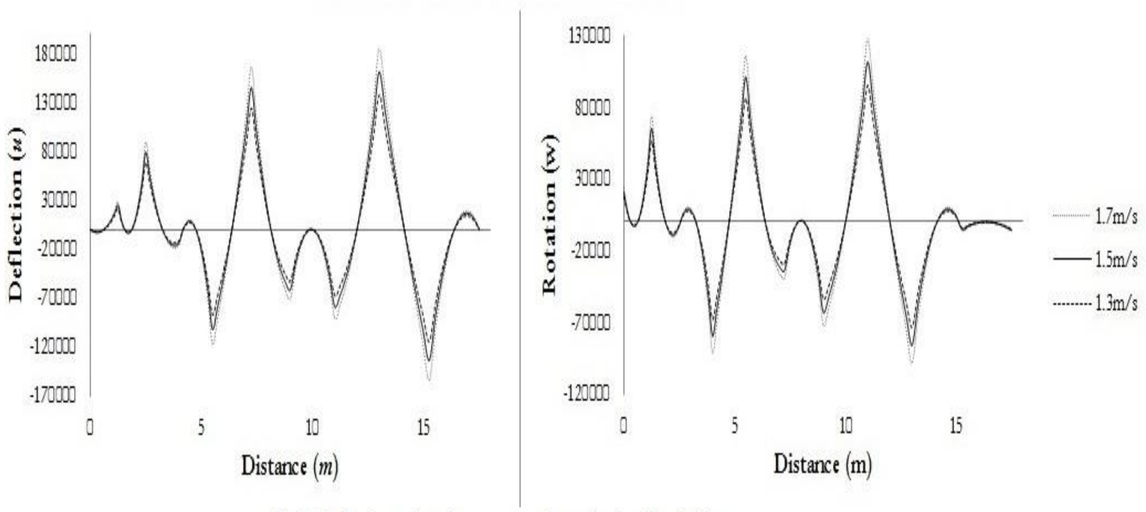

Fig. 3 : Deflection and rotation responses for various load's velocities.

4.3 Effect of the load's length on the dynamic response

To investigate the effect of the moving load's length on the dynamic response of the beam, the load’s length was increased from 0.2m to 0.4m. The dynamics response of the beam decreased as the load’s length increased as Figure4shows.

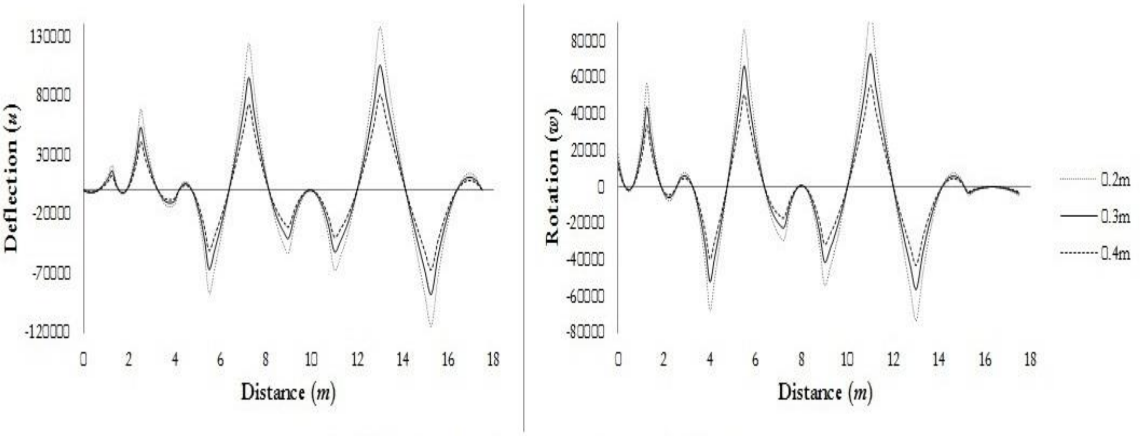

Fig. 4: Deflection and rotation responses for various load's lengths.

4.4 Effect of changes in boundary conditions on the dynamic response

The maximum amplitude of deflection *u* and rotation *w* are far higher with the simply supported boundary than for both the cantilever and clamped boundaries. As seen in Figures 5, the values for simply supported boundary are ten-thousandth compared with the actual values of the cantilever and clamped boundaries.

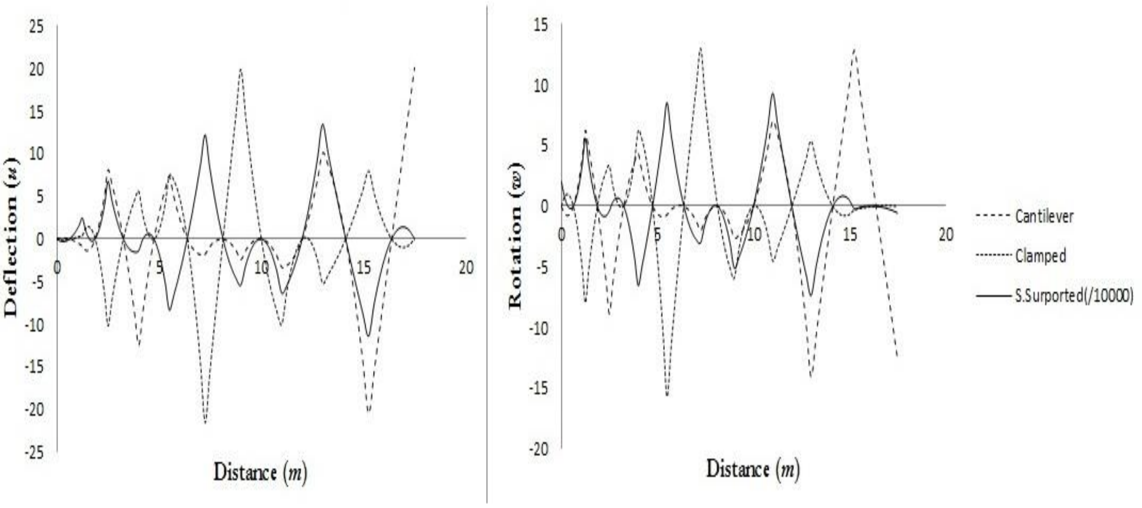

Fig. 5: Deflection and rotation responses for various boundary conditions.

5. CONCLUSION

The dynamic response of prestressed tapered Timoshenko beam was investigated in this paper. We have shown that there exists a gradual decrease in the dynamic response of the beam as the prestress is increased. This implies that a higher magnitude of prestress reduces the vibration of the structure, and hence, may improve its durability. We have also shown that other characteristics such as load's velocity and length, as well as beam's boundary conditions, also affect the dynamic response of the present beam.

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Appendix

1. The Lagrange quadratic approximation functions

$$\Phi_1(x)=\left(1-\frac{3x}{l}+\frac{2x^2}{l^2}\right),\ \Phi_2(x)=\left(\frac{4x}{l}-\frac{4x^2}{l^2}\right),\ \Phi_3(x)=\left(-\frac{x}{l}+\frac{2x^2}{l^2}\right) \tag{21}$$

$$\Psi_1(x)=\left(1-\frac{3x}{l}+\frac{2x^2}{l^2}\right),\ \Psi_2(x)=\left(\frac{4x}{l}-\frac{4x^2}{l^2}\right),\ \Psi_3(x)=\left(-\frac{x}{l}+\frac{2x^2}{l^2}\right) \tag{22}$$

2. Element stiffness, mass, and centripetal matrices, and the element applied and internal forces vector.

$$K_{ij}^{*}=\int_0^{l_e} kGA\Phi_i'\Phi_j'dx;\ K_{ij}^{\circ}=\int_0^{l_e} p_w\Phi_i'\Phi_j'dx \tag{23}$$

$$M_{ij}^{*}=\int_0^{l_e} \rho A\Phi_i\Phi_j dx\ ;\ M_{ij}^{**}=\frac{p}{\varepsilon}\int_{\xi-\frac{\varepsilon}{2}}^{\xi+\frac{\varepsilon}{2}} \Phi_i\Phi_j dx \tag{24}$$

$$C_{ij}=\frac{2pv}{\varepsilon}\int_{\xi-\frac{\varepsilon}{2}}^{\xi+\frac{\varepsilon}{2}} \Phi_i\Phi_j' dx \tag{25}$$

$$K_{ij}^{**}=\frac{pv^2}{\varepsilon}\int_{\xi-\frac{\varepsilon}{2}}^{\xi+\frac{\varepsilon}{2}} \Phi_i\Phi_j'' dx\ ;\ K_{ik}=\int_0^{l_e} kGA\Phi_i'\Psi_k dx \tag{26}$$

$$F_i^{1}=\frac{pg}{\varepsilon}\int_{\xi-\frac{\varepsilon}{2}}^{\xi+\frac{\varepsilon}{2}} \Phi_i dx+Q_i\ ; \tag{27}$$