Non-derangements in orientation reversing mappings in the dihedral group

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Abstract

Let ORD_n *be the set of orientation reversing mappings in the dihedral group* D_n. *We define a reflection* $\rho_m \in ORD_n$ *by* $\rho_m(i) \to n-i-m+1 \ (i \in X_n)$ *and we show that for* $m-$ *even and* $n-$ *odd, there is only one fixed point in* $\mathrm{ORD_n}$ *and for* $m-$ *odd and* $n-$ *even, there are exactly* $\frac{n}{2}$ *permutations each having exactly two fixed points.*

Key Words: *Permutation, derangement, symmetries, full transformation, partial transformation, partial one-one transformation*.

1. INTRODUCTION

Let X_n denote the set $\{1, 2, \cdots, n\}$ considered with standard ordering and let T_n, P_n, O_n and I_n be the full transformation semigroup, the partial transformation semigroup, the submonoid of T_n consisting of all order preserving mappings of X_n, and the semigroup of all injective partial transformations respectively. Another closely related algebraic structure to O_n and I_n are S_n and D_n the symmetric and dihedral groups on the set X_n, respectively.

In this paper, we consider the subgroup of orientation reversing bijective mappings in D_n. In particular; we shall pay attention to a subset of the Dihedral group $D_n := \{x, y \mid x^n = y^2 = 1 \quad xy = x^{-1}y\}$ of order $2n$. We denote the subsets by $ORD_n (OPD_n)$ the set of all orientation reversing (preserving) bijective mappings of n-element set in D_n.

Combinatorial properties of T_n and S_n, and some of their sub-semigroups and subgroups respectively, have been studied over a long period and many interesting and delightful results have emerged (see for example [1], [2], [4]) and those recently inspired by the works of Bashir and Umar [4] and Catarino & Higgins[8]. Bashir [3] has shown that for n-odd, there are n-even derangements, and for n-even, there are $\frac{n}{2}$ even and $\frac{n}{2}$ odd derangements, respectively, in OPD_n (the subgroup of all orientation preserving bijective mappings of n-element set).

Let X_n be the set of well-ordered elements, we define a mapping $\alpha : X_n \to X_n$ as order decreasing if $x\alpha \le x$, for all x in X_n. If $x \le y \Rightarrow x\alpha \le y\alpha$, then α is said to be order preserving for all x, y in X_n. Let $A = (a_1, a_2, \ldots, a_s)$ be a finite sequence from the chain X_n. We say that A is cyclic or has clockwise orientation if there exist not more than one subscript j such that $a_j > a_{j+1}$ where a_{s+1} denotes a_1. We say that $A = (a_1, a_2, \ldots, a_s)$ is anti-cyclic or has anticlockwise orientation if there exists no more than one subscript j such that $a_j < a_{j+1}$. Note that a sequence A is cyclic if and only if A is empty or there exist $j \in \{0, 1, \ldots, s-1\}$ such that $a_{j+1} \le a_{j+2} \le \cdots \le a_s \le a_1 \le \cdots \le a_j$. j is unique unless the sequence is a constant. Let $\alpha \in T_n$, we say α is an orientation-reversing mapping on X_n if the sequence $(1\alpha, 2\alpha, \cdots, n\alpha)$ is anti-cyclic. The collection of all orientation-reversing mappings on X_n is denoted by OR_n.

Let $\gamma \in OR_n$, γ is a reflection where by $i \to n + i - 1 (i \in X)$, and $(1\gamma, 2\gamma, \ldots, n\gamma) = (n, n-1, \ldots, 1)$ for $\gamma \in OR_n$ is anti-cyclic.

We gather some known results that we shall need in the proof of the main result.

Theorem 1.1, (Theorem 5.9,[8]) For $t \ge 3$ the maximal subgroups of $D_t^{o_n}$ are the dihedral groups of order $2t$.

Result 1.2[8] Let A be any cyclic (anti-cyclic) sequence. Then A is anti-cyclic (cyclic) if and only if A has no more than two distinct values.

If $A = (a_1, a_2, \ldots, a_t)$ is any sequence then we denote by A^τ sequence $(a_t, a_{t-1}, \ldots, a_1)$, called the reversed sequence of A.

Result 1.3[8] Let $A = (a_1, a_2, \ldots, a_t)$ be any sequence from X_n. Then A is cyclic (anti-cyclic) if and only if A^τ is anti-cyclic (cyclic).

Result 1.4[8] If $(a_1, a_2 \ldots, a_t)$ is cyclic (anti-cyclic) then so is

(a) the sequence $(a_{i_1}, a_{i_2}, \ldots, a_{i_r})$ $(i_1 < i_2 < \cdots < i_r)$

(b) and the sequence $(a_j, a_{j+1}, \ldots a_t, a_1, \ldots, a_{j-1})$, for all $1 \le j \le t$.

Result 1.5[8] Any restriction of a member of OP_n (OR_n) is also a member of OP_n (OR_n).

In what follows, we shall require that; $m, n, and\ k \in \mathrm{N}$, $n > m > k$, $m = 2k$, $0 \le k \le \frac{m}{2}$, and $0 \le m \le n-2$, If $m = 2k+1$ then $0 \le k \le \frac{m+1}{2}$ and $0 \le m \le n-2$.

Let $\rho \in D_n$, we say ρ is an orientation-reversing bijective mapping in D_n. If the sequence $(1\rho, 2\rho, \cdots, n\rho)$ is anti-cyclic, the collection of all such mappings is denoted by ORD_n.

We define a reflection $\rho_m \in ORD_n$ by $\rho_m(i) \to n-i-m+1 \ (i \in X_n)$ (or simply $(i) \to n-i-m+1 \ (i \in X_n)$) with $\rho = \rho_0 = i \to n+1-i$ such that $(1\rho , 2\rho, \ \cdots, \ n\rho) = (n, \ n-1, \ \cdots, \ 1)$ and is anti-cyclic. Thus, for every $\alpha^m \in OPD_n$ there exist an equivalence $\alpha^m\rho$ in ORD_n. That is there exist an isomorphism between OPD_n and ORD_n given by

$$\rho_m \in ORD_n, \ \ \rho_m = \alpha^m \rho = \prod_{i=1}^{\frac{n+1}{2}} (i, n-m-i+1)$$

Recall also that, a permutation σ is said to be a derangement, that is, a permutation without fixed points if $\sigma(x) \neq x$, otherwise $(if\ \sigma(x) = x)$, is the said to have a fixed point.

2. Orientation Reversing Mappings in the Dihedral Group D_n

We now give the algebraic proof of the main result (lemma 2.1 and 2.2) .

Lemma 2.1 If $n=2k+1$, we consider two cases of $\rho_m = \alpha^m \rho$

(i) If $m = 2k$, ρ_m has a fixed point at $i = \frac{n+1}{2} - \frac{m}{2}$

(ii) If $m=2k+1$ then $\rho_m \ (0 \le m \le n-1)$ has a fixed point at $i = n - \frac{m+1}{2}$

Proof: -(i). We prove the assertions by induction on m, there are several cases to examine. First recall that for all $\rho_m \in ORD_n$, $\rho_m = \alpha^m \rho = \prod_{i=1}^{\frac{n+1}{2}} (i, n-m-i+1)$

Case i: $m = 0, (m \equiv (0 mod\ n))$, $\rho = \rho_0 = \prod_{i=1}^{\frac{n+1}{2}} (i, \ n-i+1)$

The fixed point is at $0 \to i = \frac{1}{2}(n-0+1) \Rightarrow i = \frac{1}{2}(n-m+1), \ m = 0$

$$\rho_0 = (1, \ n)(2, n-1) \cdots \left(\frac{n-1}{2}, \ \frac{n+3}{2}\right)\left(\frac{n+1}{2}, \ \frac{n+1}{2}\right)$$

Case ii: We now assume that the result holds for all values of m up to $2k$. $\rho_{2k} = (i, n-i-2k+1)$.

$$\rho_{2k} = (1, \ n-2k)(2 \ , n-2k-1) \cdots \left(\frac{n-2k-1}{2}, \ \frac{n-2k+3}{2}\right)\left(\frac{n-2k+1}{2}, \ \frac{n-2k+1}{2}\right)$$

$$\cdots \left(\frac{n-1}{2}, \frac{n-2(2k-1)+1}{2}\right)\left(\frac{n+1}{2}, \ \frac{n-2(2k-1)-1}{2}\right) \cdots (n \ , \ n-2k+1)$$

ρ_{2k} has a fixed point at $2k \to i = \frac{1}{2}(n-2k+1) \Rightarrow i = \frac{1}{2}(n-m+1), \quad m = 2k.$

Case iii: finally, we consider the next even natural number after $2k$, $m = 2(k+1)$.

$$\rho_{2(k+1)} = (i,\ n-i-2k-1)$$

$$\rho_{2(k+1)} = (1,\ n-2(k+1))(2,\ n-2(k+1)-1) \cdots \left(\frac{n-2(k+1)+1}{2},\ \frac{n-2(k+1)+1}{2}\right)$$

$$\left(\frac{n-2(k+1)+3}{2},\ \frac{n-2(k+1)-1}{2}\right) \cdots \left(\frac{n-1}{2},\ \frac{n-2(2(k+1))+3}{2}\right)\left(\frac{n+1}{2},\ \frac{n-2(2(k+1))+1}{2}\right)$$

The fixed point is at $i = \frac{1}{2}(n-2(k+1)+1) \Rightarrow i = \frac{1}{2}(n-m+1), m = 2(k+1)$

The result is true for $m = 2(k+1)$, hence it is true for all $m-$ even and $n-$odd.

(ii). By a similar argument as in (i) above. we consider $m,\ n = 2k+1$.

$$\rho_m = \alpha^m \rho = \prod_{i=1}^{\frac{n+1}{2}} (i, n-m-i+1).$$

Case i: $m=1,\ n-odd$. $\rho_1 = \prod_{i=1}^{\frac{n-2}{2}} (i,\ n-i)$

The fixed point is at $i = \frac{1}{2}(n-1+1) \quad \Rightarrow i = \frac{1}{2}(n-m+1), \ m = 1 \quad for \quad n > m$

Since n and m are both odd natural numbers $0 \le m < n-1$ n-odd (the operation is orientation reversing mappings) we have $i = n$.

$$\rho_1 = (1, n-1)(2, n-2) \cdots \left(\frac{n-1}{2}, \frac{n+1}{2}\right)\left(\frac{n+3}{2}, \frac{n-3}{2}\right) \cdots (n-1, 1)(n, n)$$

Case ii: we assume that the result holds true for all values of $m = 2k+1,\ n-odd$.

$$\rho_{m(=2k+1)} = \alpha^{2k+1}\rho = \prod_{i=1}^{\frac{n-2}{2}} (i, n-i-2k)$$

$$\rho_{m(=2k+1)} = (1\ ,\ n-(2k+1))(2\ ,n-(2k+1)-1)\cdots\left(\frac{n-(2k+1)-1}{2},\ \frac{n-(2k+1)+3}{2}\right)\cdots$$

$$\left(n-\frac{(2k+1)-1}{2}\ ,\ n-\frac{(2k+1)+3}{2}\right)\left(n-\frac{(2k+1)+1}{2}\ ,\ n-\frac{(2k+1)+1}{2}\right)\ \cdots$$

$$(n-1\ ,\ n-(2k+1)+2)(n\ ,\ n-(2k+1)+1)$$

Similarly, for $m = 2k+1,\ \ n > m,\ n-odd$ the fixed point is at $i = n - \frac{(2k+1)+1}{2} \Rightarrow i = n - \frac{m+1}{2}$

Case iii: $n-odd$ and $m=2k+3,$ the next odd natural number after $2k+1.$

$$\rho_{m(=2k+3)} = \alpha^{2k+3}\rho = \prod_{i=1}^{\frac{n-2}{2}}(i,\ n-i-2k-2)$$

$$\rho_m = \alpha^{2k+3}\rho = (1\ ,\ n-(2k+3))(2\ ,\ n-(2k+3)-1)\cdots\left(\frac{n-(2k+3)-1}{2}\ ,\ \frac{n-(2k+3)+3}{2}\right)$$

$$\left(n-\frac{(2k+3)-1}{2}\ ,\ n-\frac{(2k+3)+3}{2}\right)\left(n-\frac{(2k+3)+1}{2}\ ,\ n-\frac{(2k+3)+1}{2}\right)\cdots$$

$$(n-1\ ,\ n-(2k+3)+2)(n\ ,\ n-(2k+3)+1)$$

by a similar argument as in cases I & II the fixed point is at

$$i = n - \frac{(2k+3)+1}{2} \Rightarrow i = n - \frac{m+1}{2},\ \ m = 2k+3,\quad n > m,\ .\text{n - odd}$$

The induction process proves that the result holds true for any value of $m=2k+1.$

<u>Lemma 2.2</u> If n is even, and $m=2k+1,$ then ρ_m has two fixed points at $i = \frac{1}{2}(n-m+1)$ and $i = n - \frac{m+1}{2}$.

Proof:- If n even and $m=2k+1$ then we consider various cases of m.

Case i. $m=1,\ \rho_1=\alpha^1\rho=\prod_{i=0}^{\frac{n}{2}}(i\ ,n-i)$

$$\rho_1=(1\ ,\ n-1)(2\ ,\ n-2)\ \cdots\ \left(\frac{n-2}{2}\ ,\ \frac{n+2}{2}\right)\left(\frac{n}{2}\ ,\ \frac{n}{2}\right)\ \cdots\ (n\ ,\ n)$$

$i=\frac{n}{2}$ is the first fixed point of ρ_1 when n is even and $m=2k+1$. If one point is fixed, then we have $n-1$ elements left .since $m=2k+1$, then by a similar argument as in lemma 2.1, for n and m odd. We have a second fixed point at $n-\frac{1+1}{2}\Rightarrow i=n-\frac{m+1}{2},\ m=1.$

Similarly, if $m=3,\ \rho_3=\prod_{i=1}^{\frac{n}{2}}(i\ ,\ n-i-2)$

$$\rho_3=\alpha^3\rho=(1\ ,\ n-3)(2\ ,n-4)\cdots\left(\frac{n-2}{2}\ ,\ \frac{n-2}{2}\right)\left(\frac{n-4}{2}\ ,\ \frac{n}{2}\right)\ \cdots\ (n-1\ ,\ n-1)(n\ ,\ n-2)$$

$i=\frac{1}{2}(n-3+1)$ is a fixed point.

By a similar argument as in case I above, the other fixed point is at

$$i=n-\frac{3+1}{2}=n-1\Rightarrow i=n-\frac{m+1}{2},\quad m=3.$$

Case ii: Let's assume that the result is true for all values of m up to $m=2k+1$.

$$\rho_{2k+1}=\prod_{i=1}^{\frac{n}{2}}(i\ ,\ n-i-2k)$$

$$\rho_{2k+1}=(1,\ \ n-(2k+1))(2\ ,\ n-(2k+1)+1)\ \cdots\ \left(\frac{n-(2k+1)-1}{2}\ ,\ \frac{n-(2k+1)+3}{2}\right)$$

$$\left(\frac{n-(2k+1)+1}{2}\ ,\ \frac{n-(2k+1)+1}{2}\right)\ \cdots\ \left(n\ -\ \frac{(2k+1)-1}{2}\ ,\ n\ -\ \frac{(2k+1)+3}{2}\right)$$

$$\left(n\ -\ \frac{(2k+1)+1}{2}\ ,\ n\ -\ \frac{(2k+1)+1}{2}\right)\cdots(n-1\ ,\ n-(2k+1)+2)(n\ ,\ n-(2k+1)+1)$$

$$i=\frac{n-(2k+1)+1}{2}\ \Rightarrow\ i=\frac{n}{2}\ -\ \frac{m+1}{2},\quad m=2k+1\text{ is a fixed point.}$$

By similar argument as in cases i&ii above, the second fixed point is $i=n\ -\ \frac{m+1}{2}$

Case iii. $m=2k+3,$ the next odd natural number after $2k+1$

$$\rho_{2k+3} = \left(1\ ,\ n-(2k+3)\right)\left(2\ ,\ n-(2k+3)-1\right)\cdots\left(\frac{n-(2k+3)-1}{2}\ ,\ \frac{n-(2k+3)}{2}\right)$$

$$\left(\frac{n-(2k+3)+1}{2}\ ,\ \frac{n-(2k+3)+1}{2}\right)\cdots\left(n\ -\ \frac{(2k+3)-1}{2}\ ,\ n-\ \frac{(2k+3)+3}{2}\right)$$

$$\left(n\ -\ \frac{(2k+3)+1}{2}\ ,\ n\ -\ \frac{(2k+3)+1}{2}\right)\ \cdots\ \left(n-1\ ,\ \ n-2k-1\right)\left(n\ \ ,\ n-2k-2\right)$$

The first point is at $i = \dfrac{n-(2k+3)+1}{2} \Rightarrow i = \dfrac{n}{2} - \dfrac{m+1}{2},\ \ m = 2k+3.$

By a similar argument as in above cases, the second fixed point is at

$$i = n - \frac{(2k+3)+1}{2} \Rightarrow i = n - \frac{m+1}{2},\ \ m = 2k+3.$$

The induction process show that for n even and $m=2k+1$ the permutation [ρ_m has two fixed points at $i = \frac{1}{2}\left(n - m + 1\right)\ and\ \ n - \frac{m+1}{2}$

<u>Lemma 2.3</u> If n is even and $m-$odd for every $\rho \in ORD_n$ there are exactly $\frac{n}{2}$ permutations each having exactly two fixed points in ORD_n.

Proof:- If n is even ($n=4k\ or\ (4k+2)$), it is clear that there are $\frac{n}{2}$ even numbers of $m's$ in n, and $\frac{n}{2}$ odd numbers of $m's$ in n. If $m=2k+1$ there are $\frac{n}{2}$ odd $m's$ in n. It implies that there are $\frac{n}{2}$ permutations with two fixed points in n by Lemma 2.2

<u>Lemma 2.4</u> If n is odd, ρ_m has one fixed point for all m, $\rho_m \in ORD_n$.

Proof:- This result follows from Lemma 2.1

Table of number of even n permutations with k fixed pints $e(n,k)$ in Dihedral group

Table 1. $e(n,k)$

k \ n	0	1	2	3	4	5	6	7	8	9	$\sum e(n,k)$
0	1										1
1	0	1									1
2	0	0	1								1
3	2	0	0	1							3
4	3	0	0	0	1						4
5	4	5	0	0	0	1					10
6	2	0	3	0	0	0	1				6
7	6	0	0	0	0	0	0	1			7
8	7	0	0	0	0	0	0	0	1		8
9	8	9	0	0	0	0	0	0	0	1	18

Table of number of even n permutations with k fixed pints $e'(n,k)$ in Dihedral group

Table 2. $e'(n,k)$

K \ n	0	1	2	3	4	5	6	7	8	9	$\sum e'(n,k)$
0	0										0
1	0	0									0
2	0	0	0								0
3	0	3	0	0							3
4	2	0	2	0	0						4
5	0	0	0	0	0	0					0
6	6	0	0	0	0	0	0				6
7	0	7	0	0	0	0	0	0			7
8	4	0	4	0	0	0	0	0	0		8
9	0	0	0	0	0	0	0	0	0	0	0

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